Vectors & Matrices

Fundamentals

Rahul Singh
rsingh@arrsingh.com

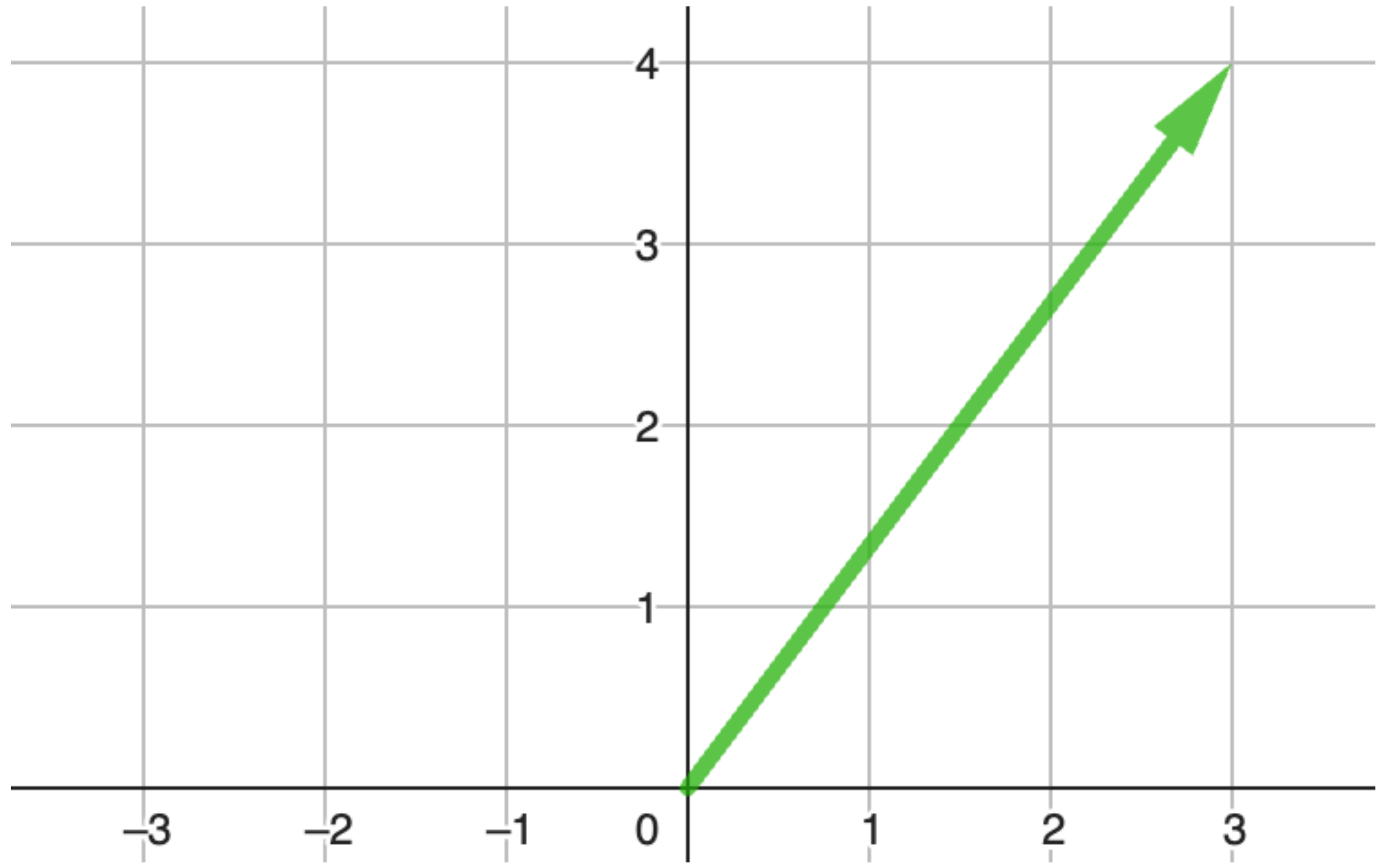
Scalars & Vectors

Scalar: A scalar is a single numeric value (positive, negative or zero)

Vector: A Vector (also known as a Euclidean Vector) is a geometric object with magnitude & direction

A Vector is represented by an ordered list of scalars - the endpoint of the vector in the cartesian coordinate system

$$v = (3,4)$$



A Vector in 2D Space

Scalars & Vectors

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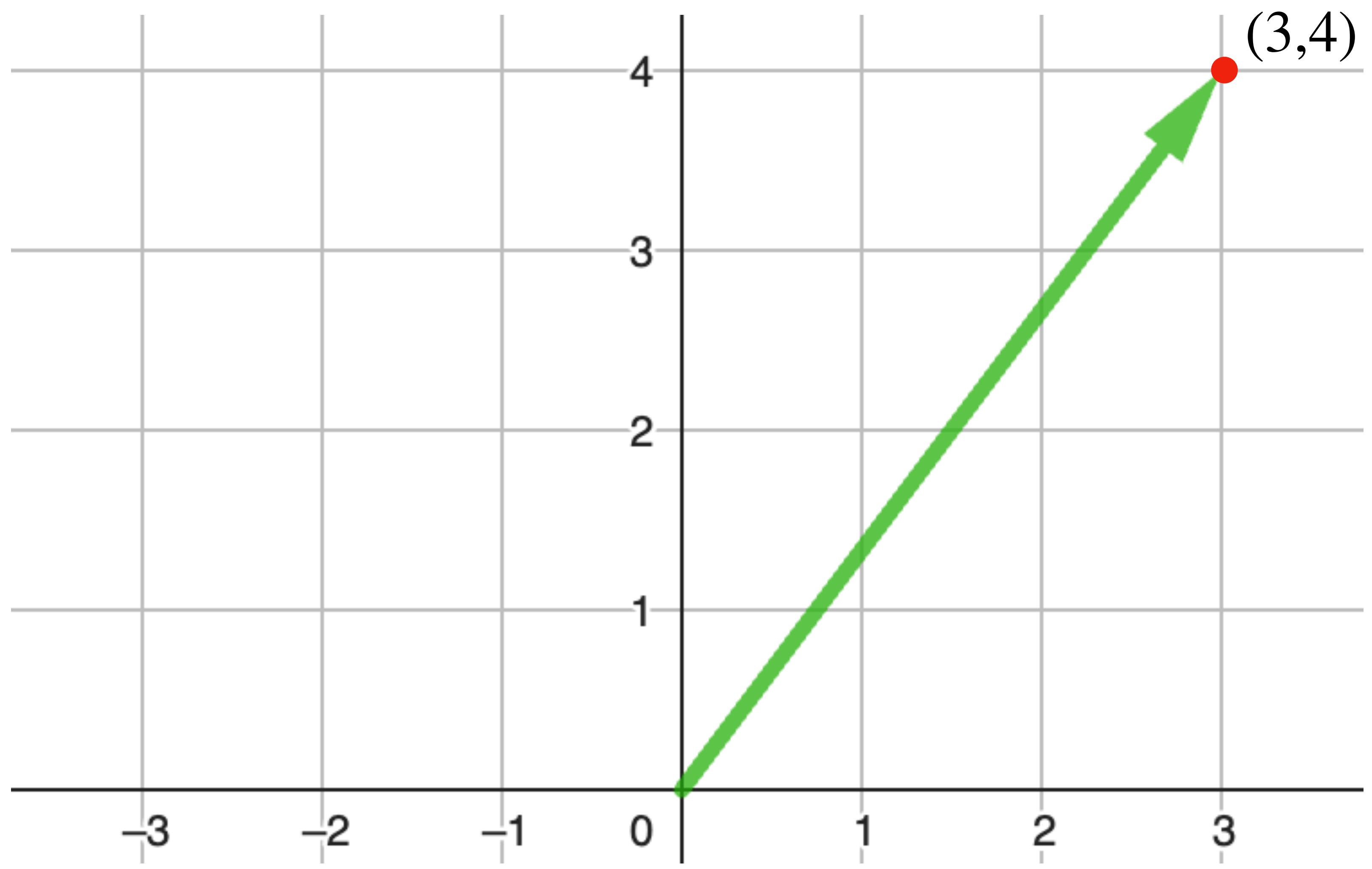
$$v = (3,4)$$

v is a vector in a Two Dimensional space

v is a vector in $\mathbf{R}^2$



A Vector in 2D Space

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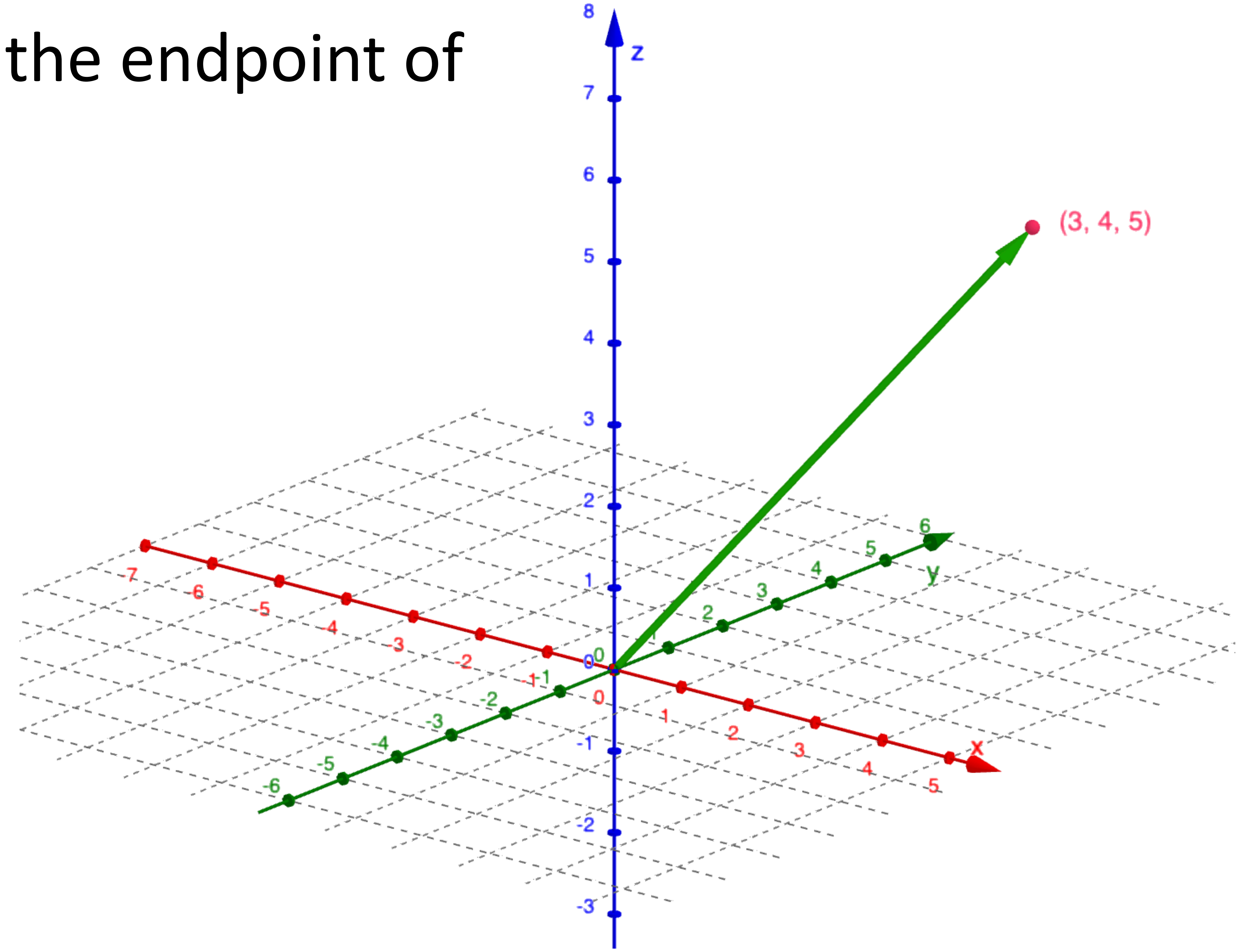
$$v = (3,4,5)$$

v is a vector in a Three Dimensional space

v is a vector in $\mathbf{R}^3$



A Vector in 3D Space

Scalars & Vectors

Scalar: A scalar is a single numeric value (positive, negative or zero)

Vector: A Vector (also known as a Euclidean Vector) is a geometric object with magnitude & direction

A Vector is represented by an ordered list of scalars - the endpoint of the vector in the cartesian coordinate system

$$v = (3,4,5,6)$$

v is a vector in a Four Dimensional space

v is a vector in $\mathbf{R}^4$

Scalar: A scalar is a single numeric value (positive, negative or zero)

Vector: A Vector (also known as a Euclidean Vector) is
a geometric object with magnitude & direction

A Vector is represented by an ordered list of scalars - the endpoint of the vector in the cartesian coordinate system

$$v = (x_1, x_2, x_3 \dots x_n)$$

The Scalar components of a Vector can be laid out as rows or columns

A Vector in n dimensional Euclidean space

v is a vector in $\mathbf{R}^n$

Lets generalize this to n dimensions

Scalars & Vectors

Scalar: A scalar is a single numeric value (positive, negative or zero)

Vector: A Vector (also known as a Euclidean Vector) is a geometric object with magnitude & direction

A Vector is represented by an ordered list of scalars - the endpoint of the vector in the cartesian coordinate system

$$v = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \dots \\ x_n \end{bmatrix}$$

v is a **column** vector in $\mathbf{R}^n$

n rows and 1 column

$$v = [x_1 \quad x_2 \quad x_3 \quad \dots \quad x_n]$$

v is a **row** vector in $\mathbf{R}^n$

1 row and n columns

Scalars & Vectors

Scalar: A scalar is a single numeric value (positive, negative or zero)

Vector: A Vector (also known as a Euclidean Vector) is a geometric object with magnitude & direction

A Vector is represented by an ordered list of scalars - the endpoint of the vector in the cartesian coordinate system

$$v = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \dots \\ x_n \end{bmatrix}$$

v is a **column** vector in $\mathbf{R}^n$

$(n \times 1)$ column vector

$$v = [x_1 \quad x_2 \quad x_3 \quad \dots \quad x_n]$$

v is a **row** vector in $\mathbf{R}^n$

$(1 \times n)$ row vector

Transpose a Vector

Scalar: A scalar is a single numeric value (positive, negative or zero)

Vector: A Vector (also known as a Euclidean Vector) is
a geometric object with magnitude & direction

Transpose is an operation that swaps the rows and columns of a Vector

$$v = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \dots \\ x_n \end{bmatrix}$$

$$v^T = [x_1 \quad x_2 \quad x_3 \quad \dots \quad x_n]$$

v is a **column** vector in $\mathbf{R}^n$

$(n \times 1)$ column vector

v^T is a **row** vector in $\mathbf{R}^n$

$(1 \times n)$ row vector

Transpose a Vector

Scalar: A scalar is a single numeric value (positive, negative or zero)

Vector: A Vector (also known as a Euclidean Vector) is a geometric object with magnitude & direction

Transpose is an operation that swaps the rows and columns of a Vector

$$v^T = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \dots \\ x_n \end{bmatrix}$$

$$v = [x_1 \quad x_2 \quad x_3 \quad \dots \quad x_n]$$

$$\boxed{(v^T)^T = v}$$

v^T is a **column** vector in $\mathbf{R}^n$

$(n \times 1)$ column vector

v is a **row** vector in $\mathbf{R}^n$

$(1 \times n)$ row vector

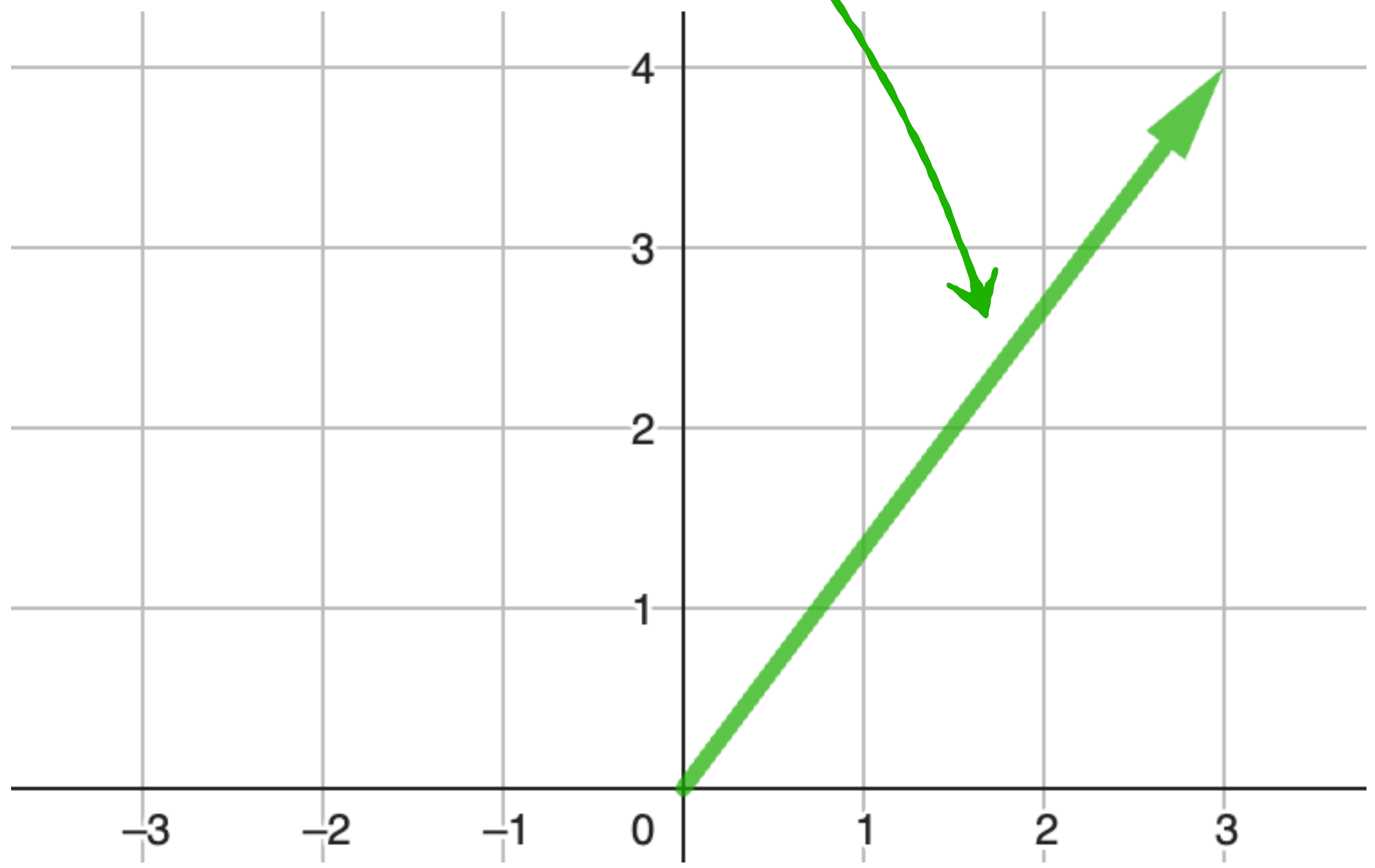
Vector: A Vector is a geometric object with **magnitude** & direction

Magnitude of a Vector is the distance between the two points.

v is a vector in $\mathbf{R}^2$

$$v = (3,4)$$

$\boxed{\text{Magnitude is the length of the Vector. The distance between } (0,0) \text{ and } (3,4)}$



A Vector in 2D Space

Vector: A Vector is a geometric object with **magnitude** & direction

Magnitude of a Vector is the distance between the two points.

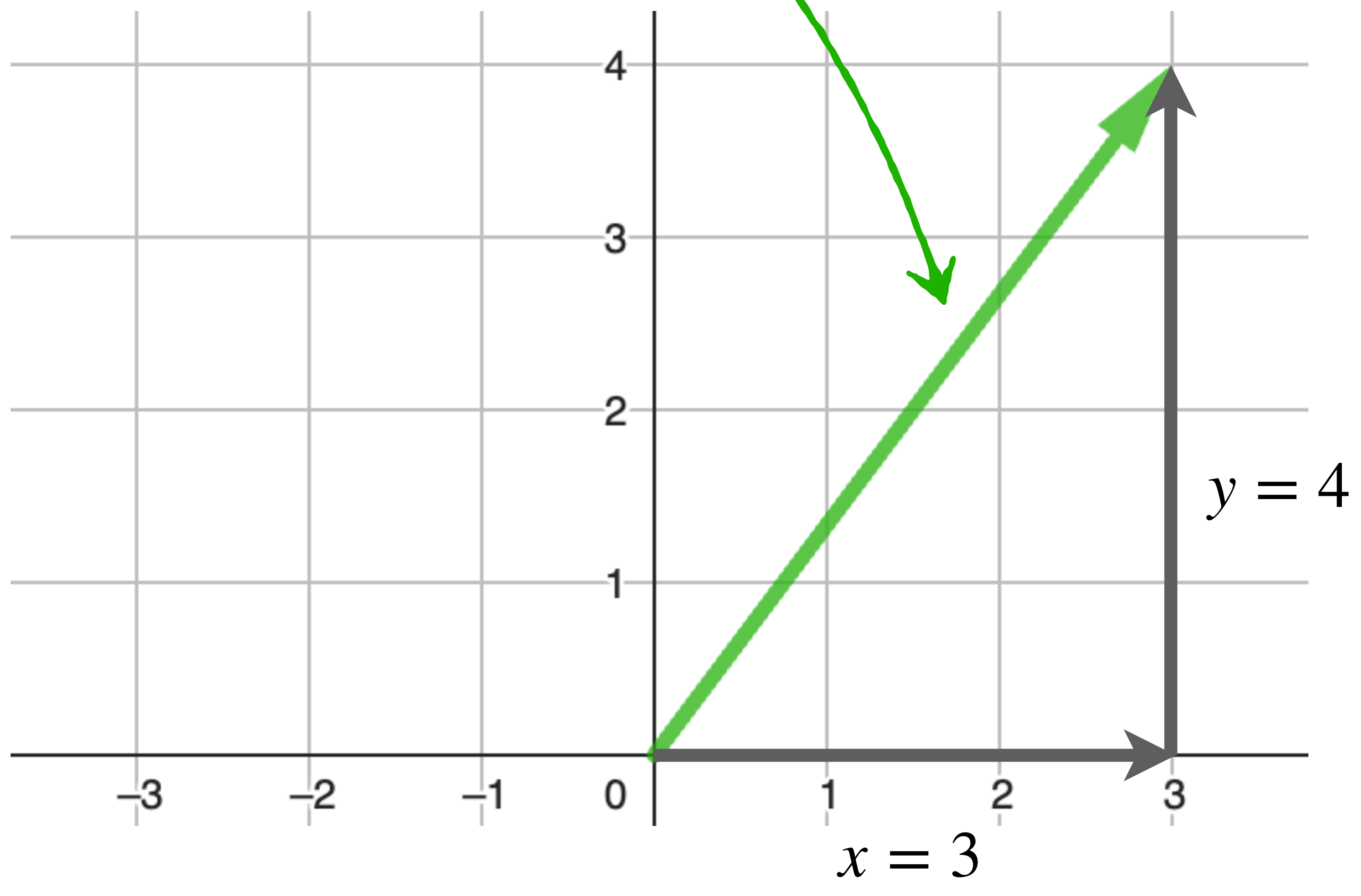
v is a vector in $\mathbf{R}^2$

$$v = (3,4)$$

Magnitude is the length of the Vector.
The distance between $(0,0)$ and $(3,4)$

Pythagoras Theorem can be used to calculate the length of the vector

$$m = \sqrt{x^2 + y^2} = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$



A Vector in 2D Space

Vector: A Vector is a geometric object with **magnitude** & direction

Magnitude of a Vector is the distance between the two points.

v is a vector in $\mathbf{R}^2$

$$v = (3,4)$$

Magnitude is the length of the Vector.
The distance between $(0,0)$ and $(3,4)$

Magnitude is also known as the Euclidean Norm of the Vector

Pythagoras Theorem can be used to calculate the length of the vector

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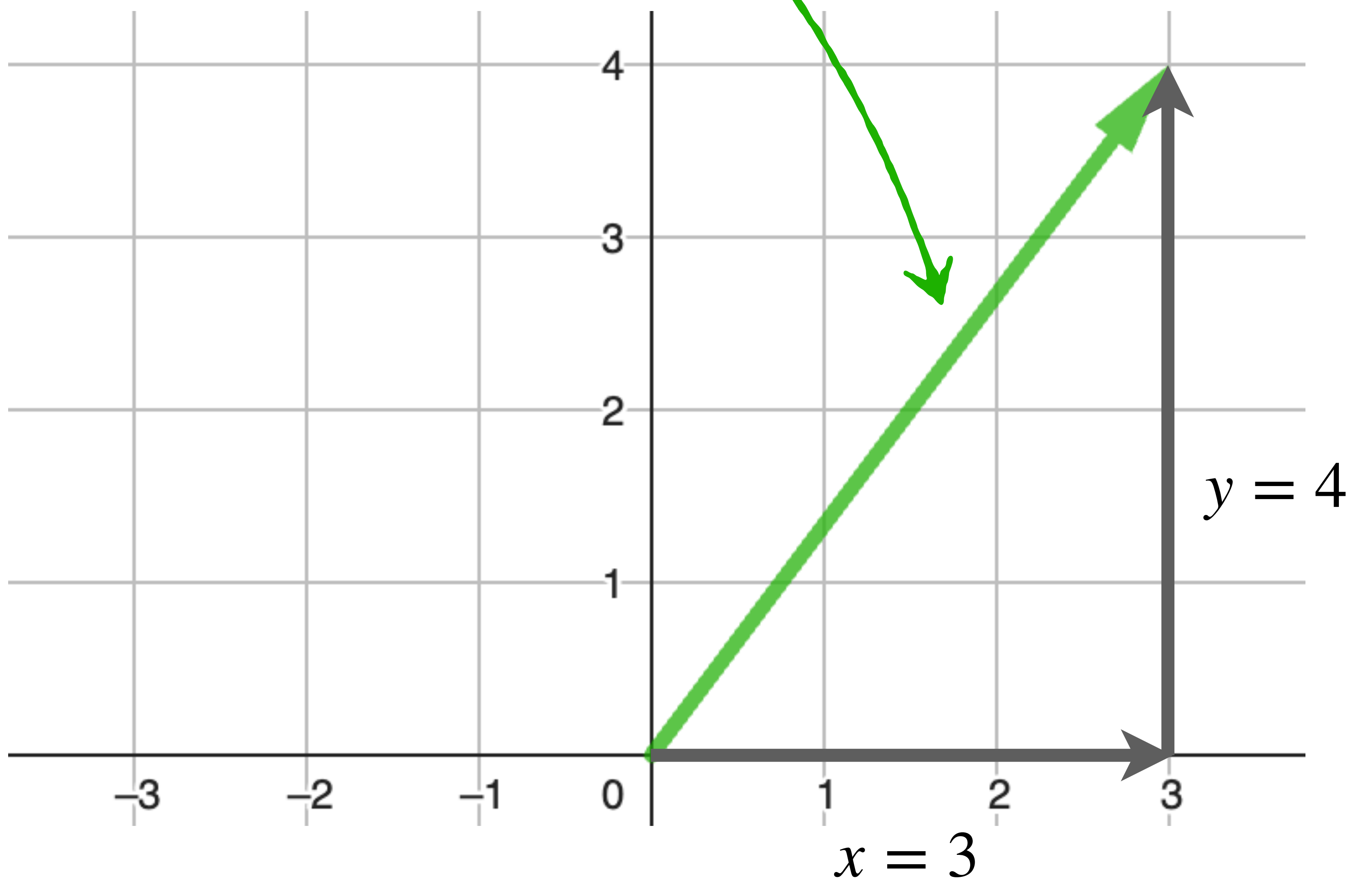


A Vector in 2D Space

Scalars & Vectors

Vector: A Vector is a geometric object with **magnitude** & direction

Magnitude of a Vector is the distance between the two points.

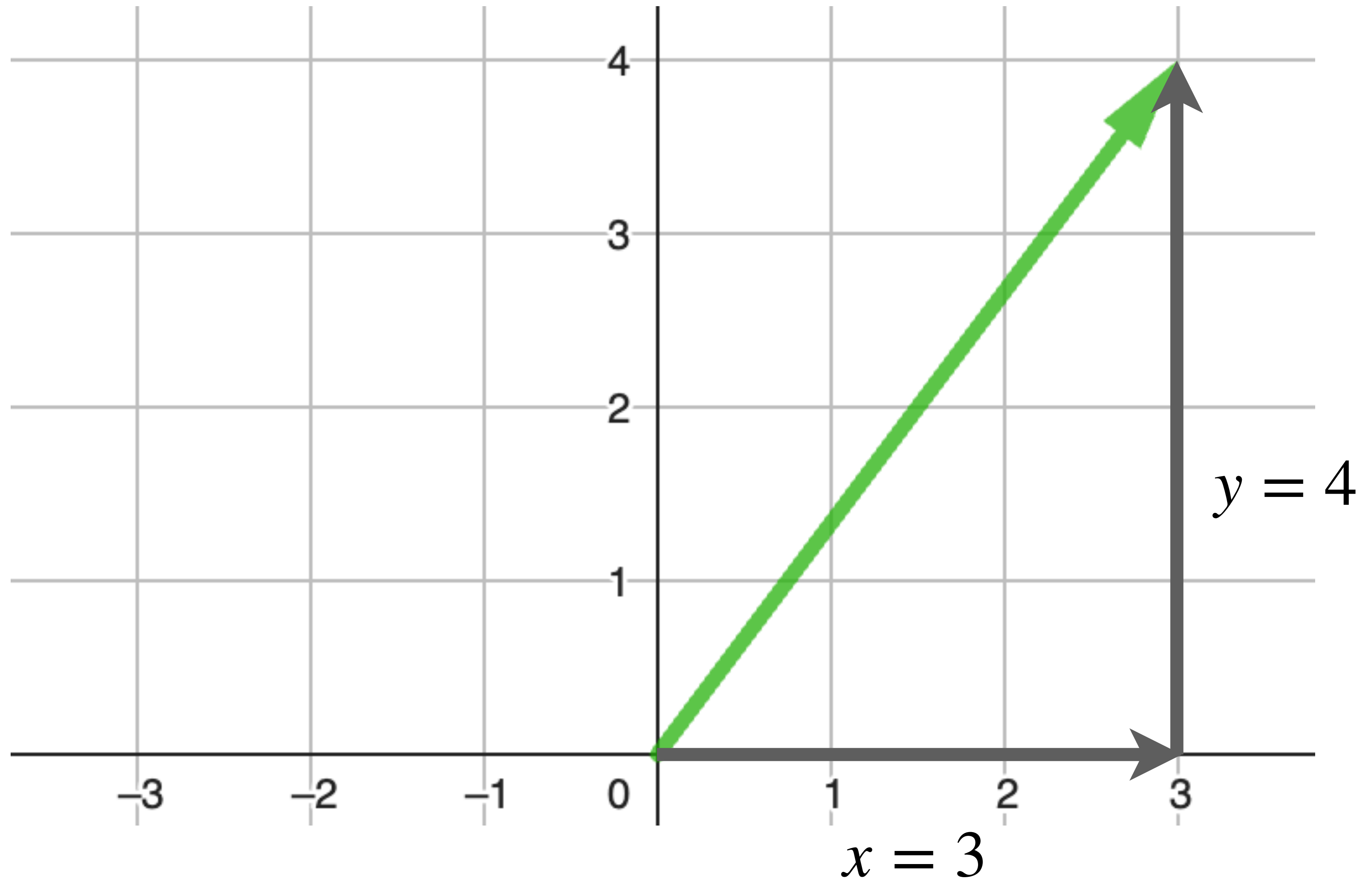
v is a vector in $\mathbf{R}^2$

$\boxed{v \text{ is a vector in a Two Dimensional space}}$

$$v = (3,4)$$

Magnitude is also known as the Euclidean Norm of the Vector

$$\| v \| = \sqrt{3^2 + 4^2} = 5$$



A Vector in 2D Space

Scalars & Vectors

Vector: A Vector is a geometric object with **magnitude** & direction

Magnitude of a Vector is the distance between the two points.

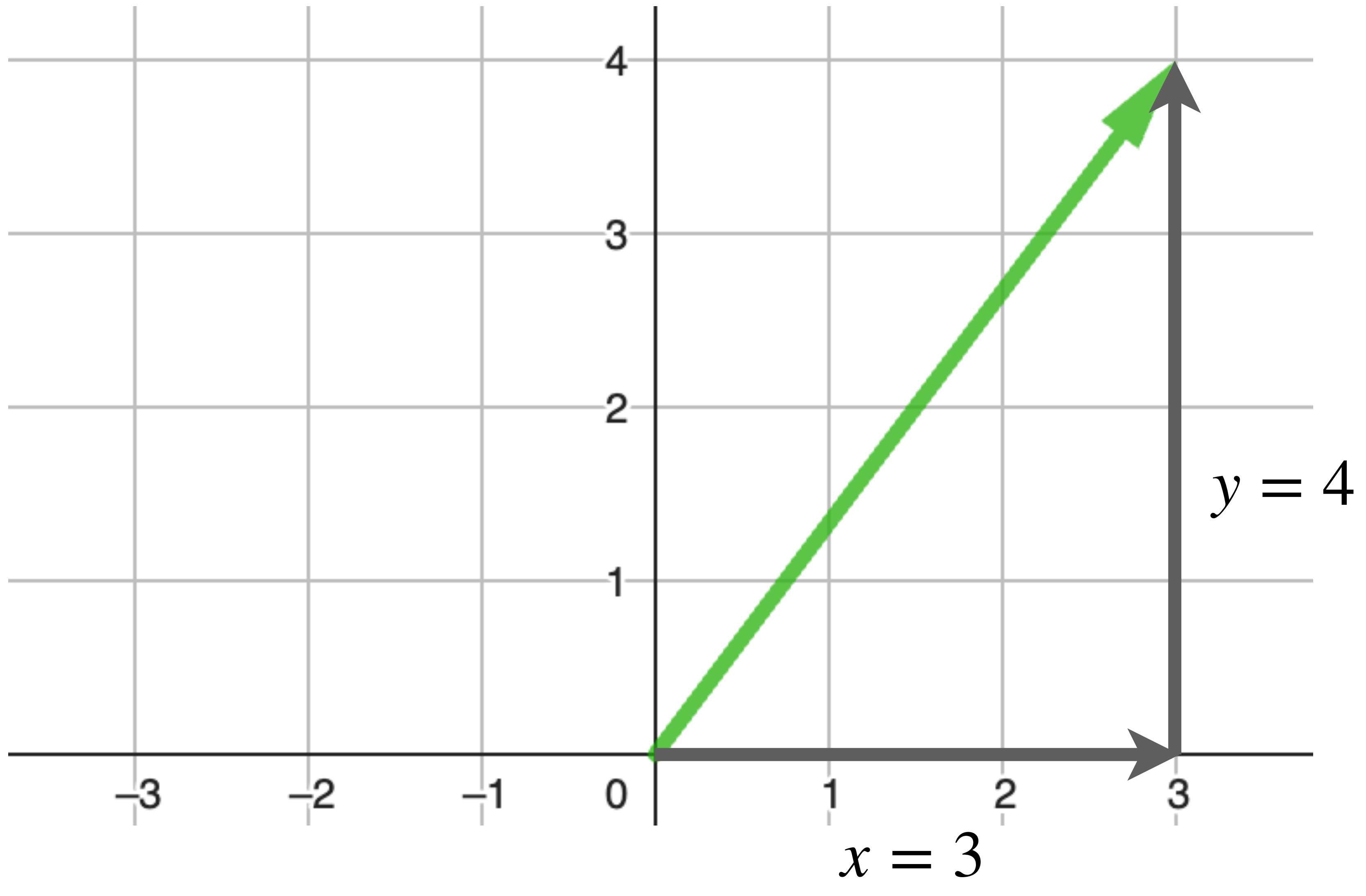
v is a vector in $\mathbf{R}^3$

v is a vector in a Three Dimensional space

$$v = (3,4,5)$$

Magnitude is also known as the Euclidean Norm of the Vector

$$\| v \| = \sqrt{3^2 + 4^2 + 5^2} = 7.07$$



A Vector in 2D Space

Vector: A Vector is a geometric object with **magnitude** & direction

Magnitude of a Vector is the distance between the two points.

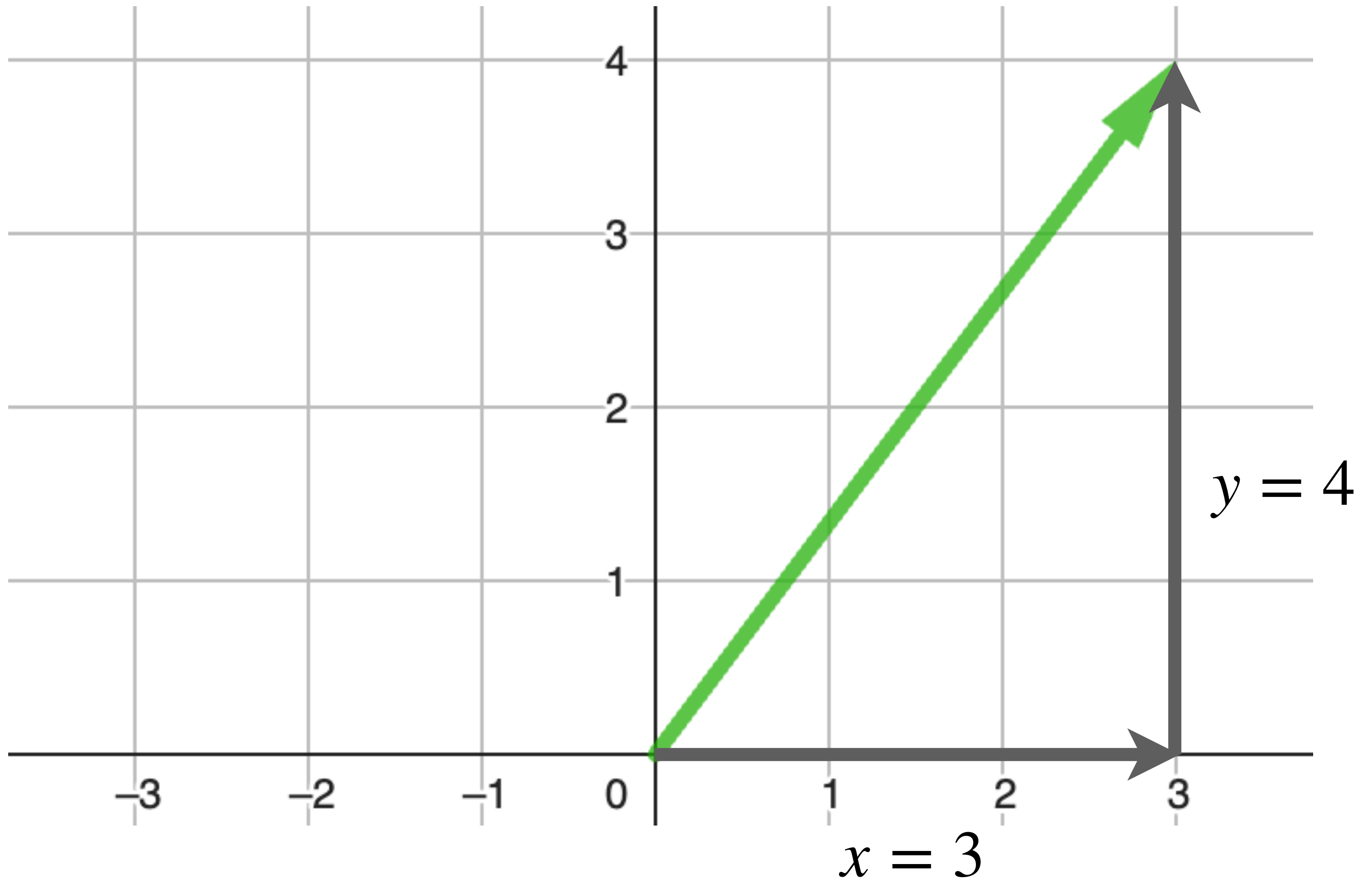
v is a vector in $\mathbf{R}^n$

$\boxed{v \text{ is a vector in a } n \text{ Dimensional space}}$

$$v = (x_1, x_2, x_3 \ldots x_n)$$

Magnitude is also known as the Euclidean Norm of the Vector

$$\| v \| = \sqrt{x_1^2 + x_2^2 + x_3^2 \ldots x_n^2}$$



A Vector in 2D Space

Unit Vector

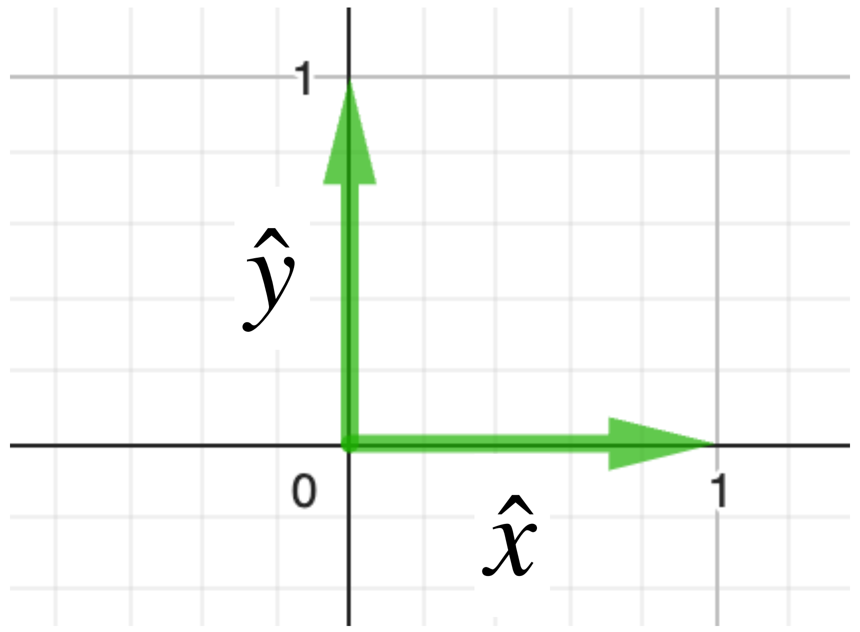
A **Unit Vector** is a vector of length 1 and is used to represent directions

$$\hat{v} = \frac{v}{\| v \|}$$

A Unit Vector (represented by the "hat") is computed by dividing a vector by its magnitude

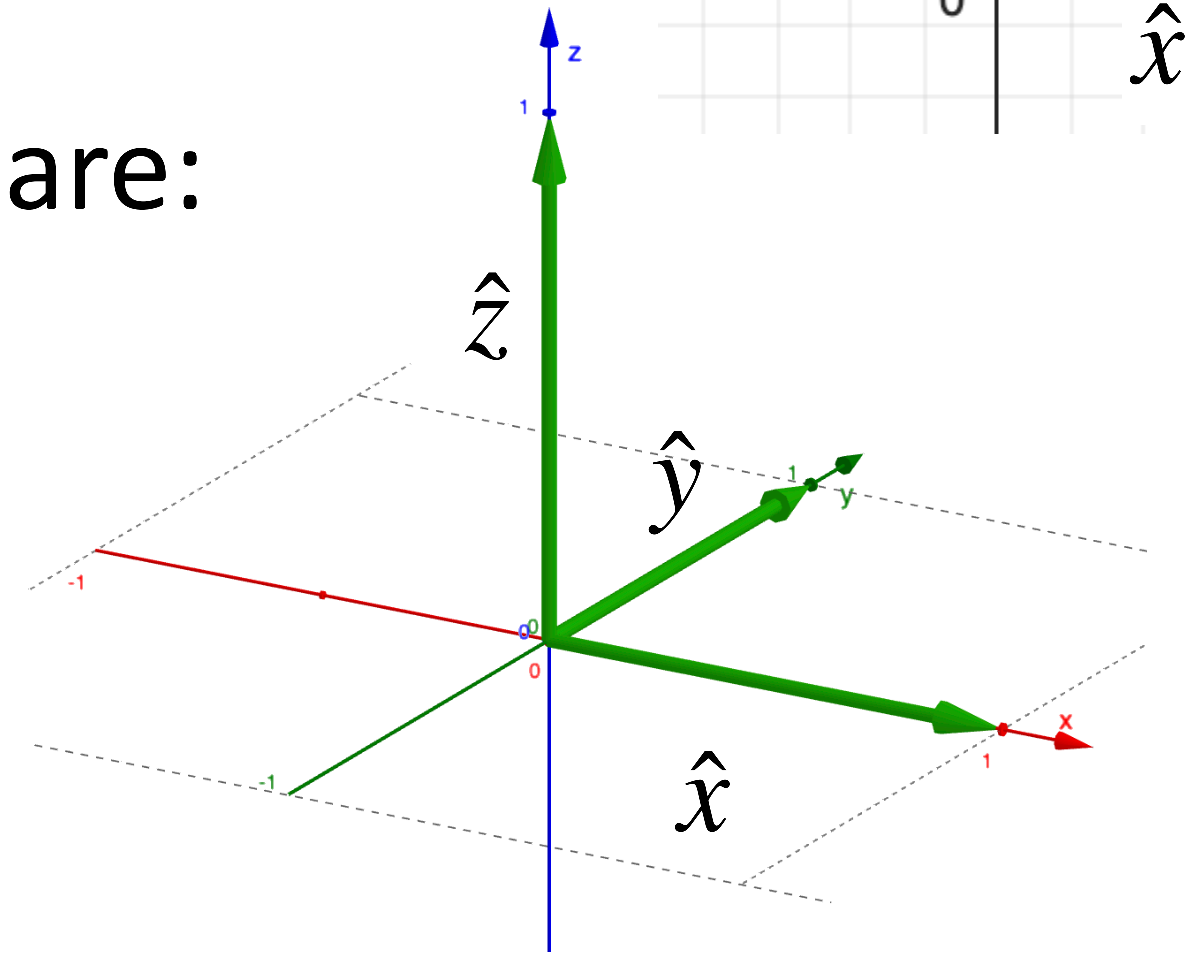
In $\mathbf{R}^2$ the unit vectors in the direction of the x and y axes are:

$$\hat{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \hat{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



In $\mathbf{R}^3$ the unit vectors in the direction of the x, y and z axes are:

$$\hat{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \hat{y} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \hat{z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$



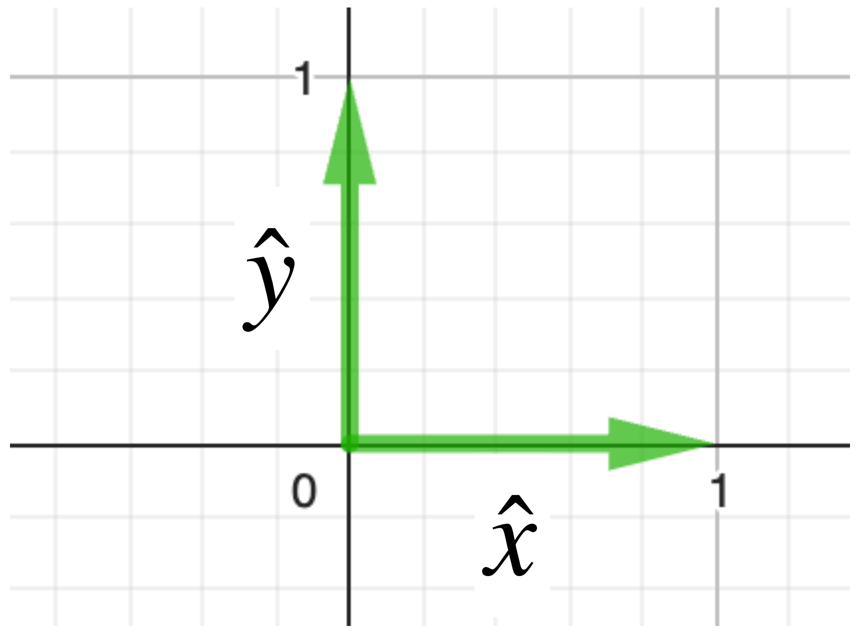
Unit Vector

A **Unit Vector** is a vector of length 1 and is used to represent directions

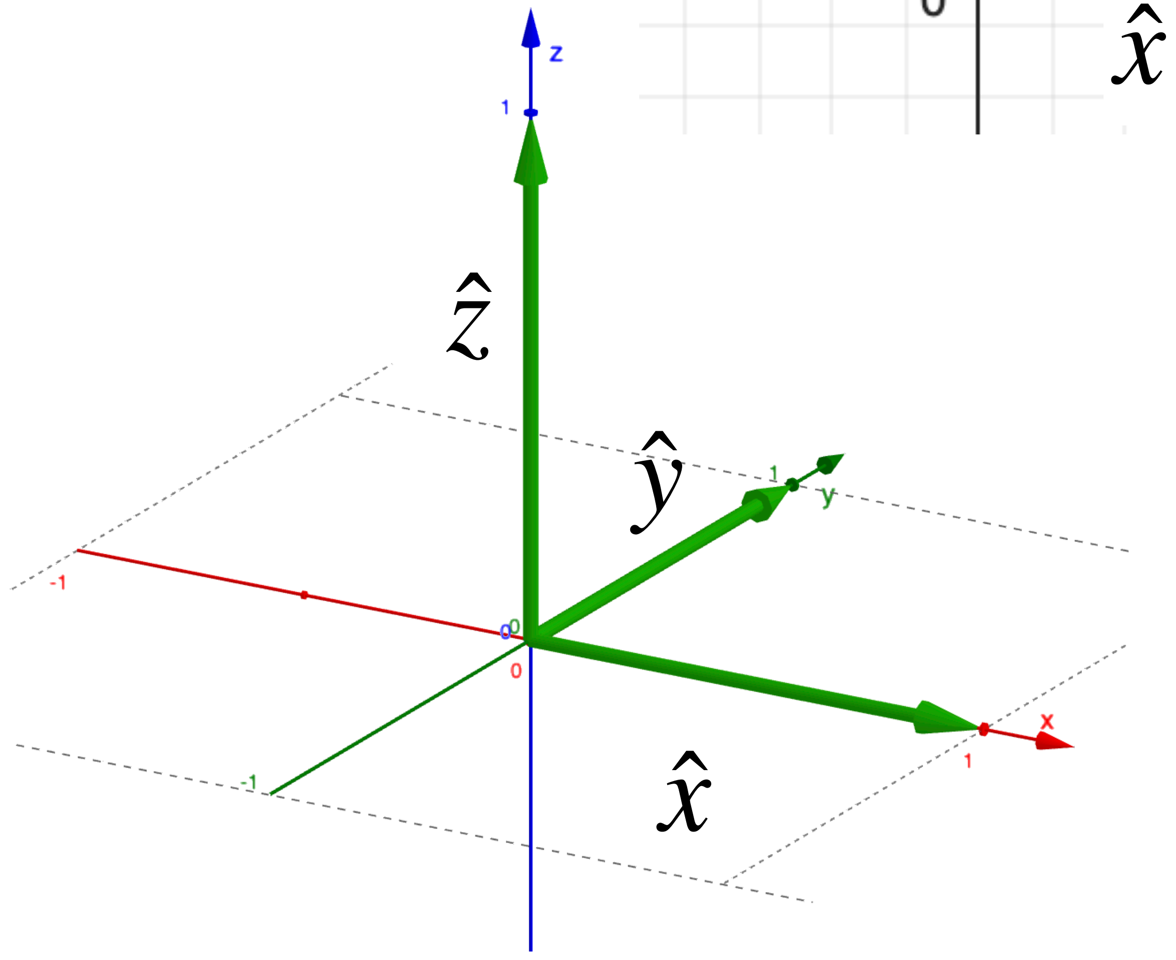
$$\hat{v} = \frac{v}{\| v \|}$$

A Unit Vector (represented by the “hat”) is computed by dividing a vector by its magnitude

Every Vector can be written as the linear combination of unit vectors



$$v = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \qquad \Rightarrow v = 2\hat{x} + 3\hat{y}$$



$$v = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \qquad \Rightarrow v = 2\hat{x} + 4\hat{y} + x\hat{z}$$

Addition of Vectors

Sum of Two Vectors is the sum of the scalar components of the Vectors

v_1 and v_2 are vectors in $\mathbf{R}^2$

$$v_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$v_1 + v_2 = \begin{bmatrix} (3+4) \\ (4+3) \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$$

Addition of Vectors

Sum of Two Vectors is the sum of the scalar components of the Vectors

v_1 and v_2 are vectors in $\mathbf{R}^n$

$$v_1 = \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \\ \dots \\ x_{1n} \end{bmatrix} \qquad v_2 = \begin{bmatrix} x_{21} \\ x_{22} \\ x_{23} \\ \dots \\ x_{2n} \end{bmatrix}$$

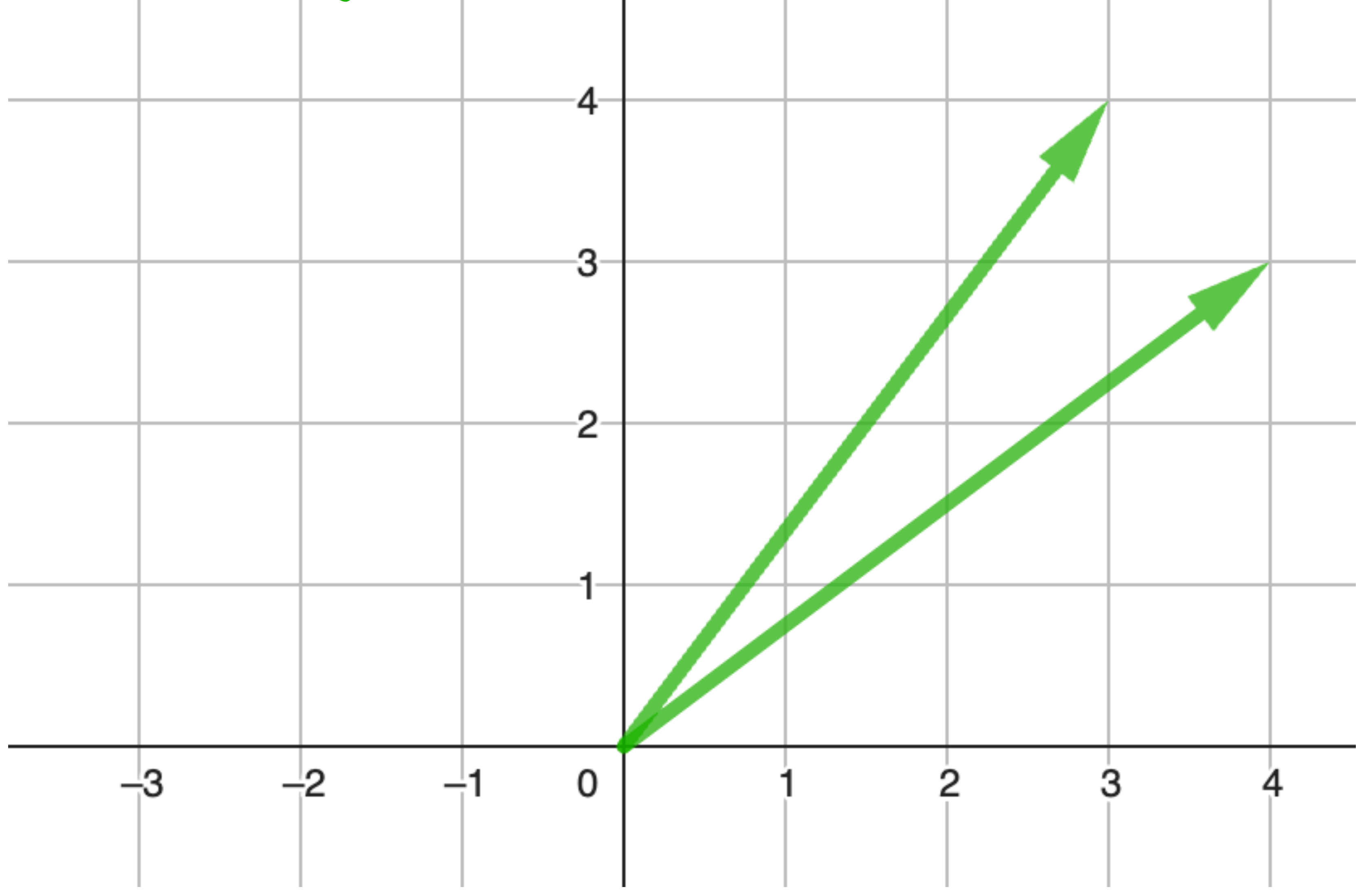
$$v_1 + v_2 = \begin{bmatrix} (x_{11} + x_{21}) \\ (x_{12} + x_{22}) \\ (x_{13} + x_{23}) \\ \dots \\ (x_{1n} + x_{2n}) \end{bmatrix}$$

Vector Addition is commutative

$v_1 + v_2 = v_2 + v_1$

Vector Addition is associative

$v_1 + (v_2 + v_3) = (v_1 + v_2) + v_3$



Subtraction of Vectors

Difference of Two Vectors is the difference of the scalar components of the Vectors

v_1 and v_2 are vectors in $\mathbf{R}^2$

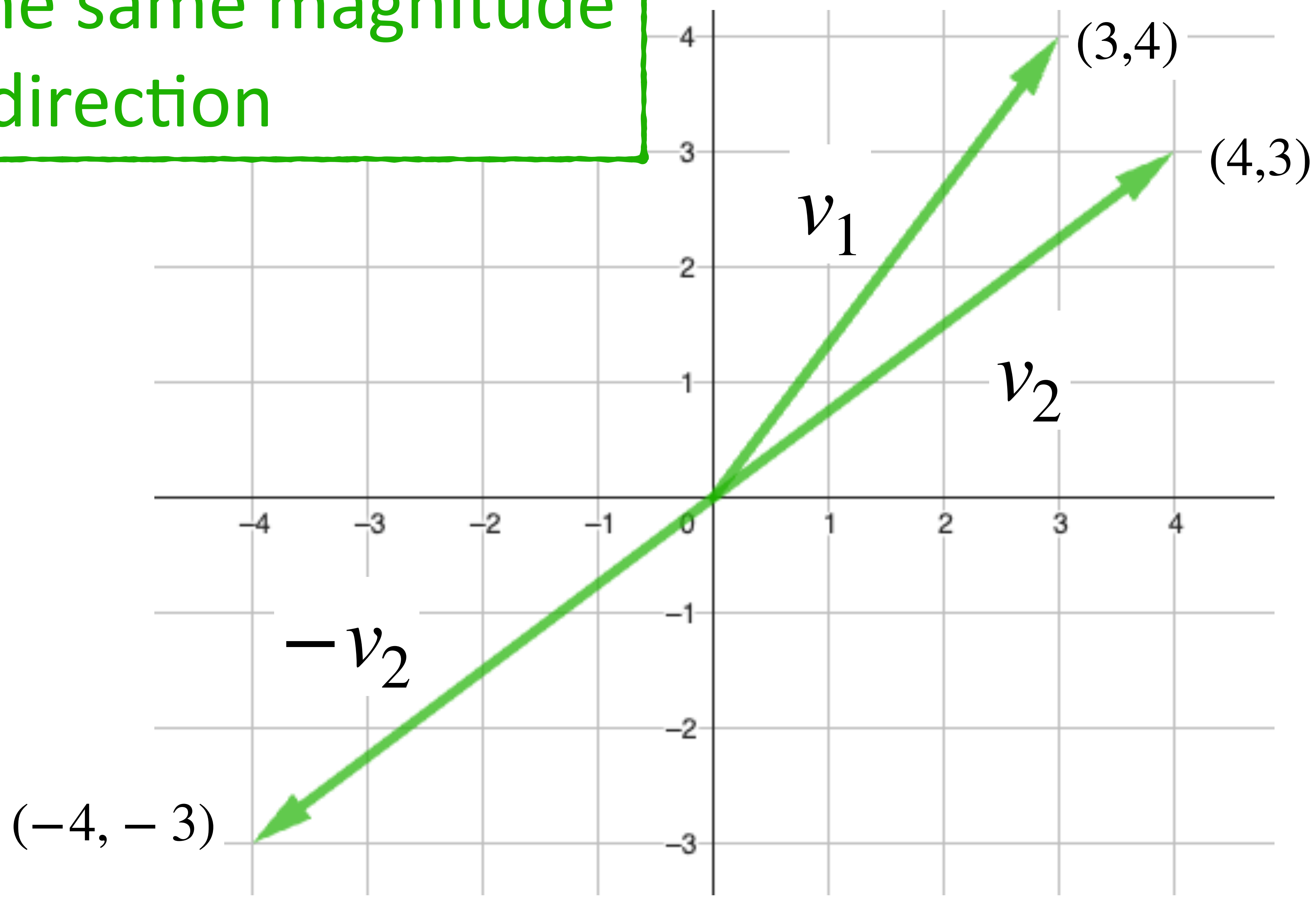
$$v_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$v_1 - v_2 = \begin{bmatrix} (3-4) \\ (4-3) \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Vector Subtraction is the same as adding a negative vector

$v_1 - v_2 = v_1 + (-v_2)$

Negative Vector is the same magnitude but in the opposite direction



Subtraction of Vectors

Difference of Two Vectors is the difference of the scalar components of the Vectors

v_1 and v_2 are vectors in $\mathbf{R}^n$

$$v_1 = \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \\ \cdots \\ x_{1n} \end{bmatrix} \qquad v_2 = \begin{bmatrix} x_{21} \\ x_{22} \\ x_{23} \\ \cdots \\ x_{2n} \end{bmatrix}$$

$$v_1 - v_2 = \begin{bmatrix} (x_{11} - x_{21}) \\ (x_{12} - x_{22}) \\ (x_{13} - x_{23}) \\ \cdots \\ (x_{1n} - x_{2n}) \end{bmatrix}$$

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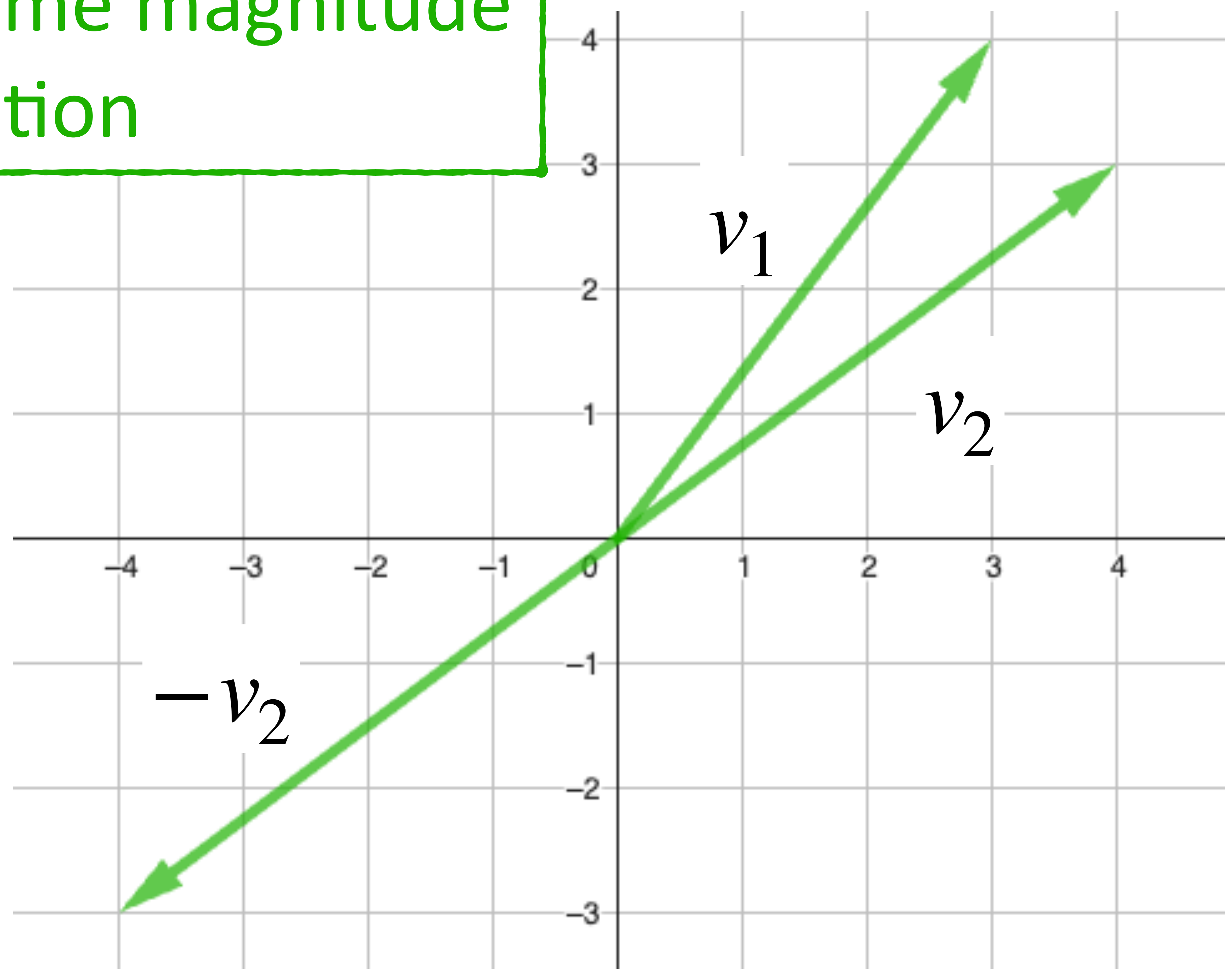
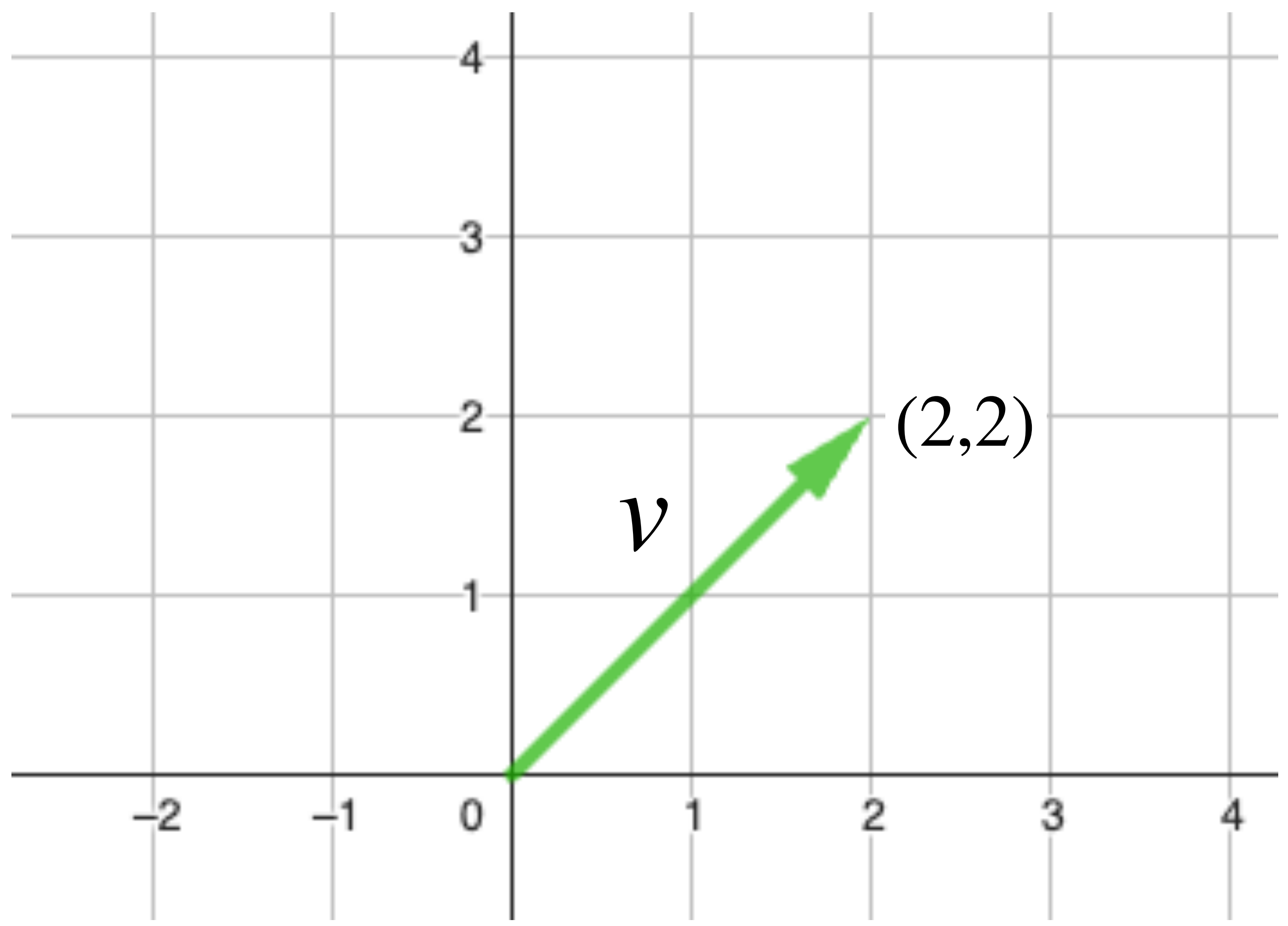


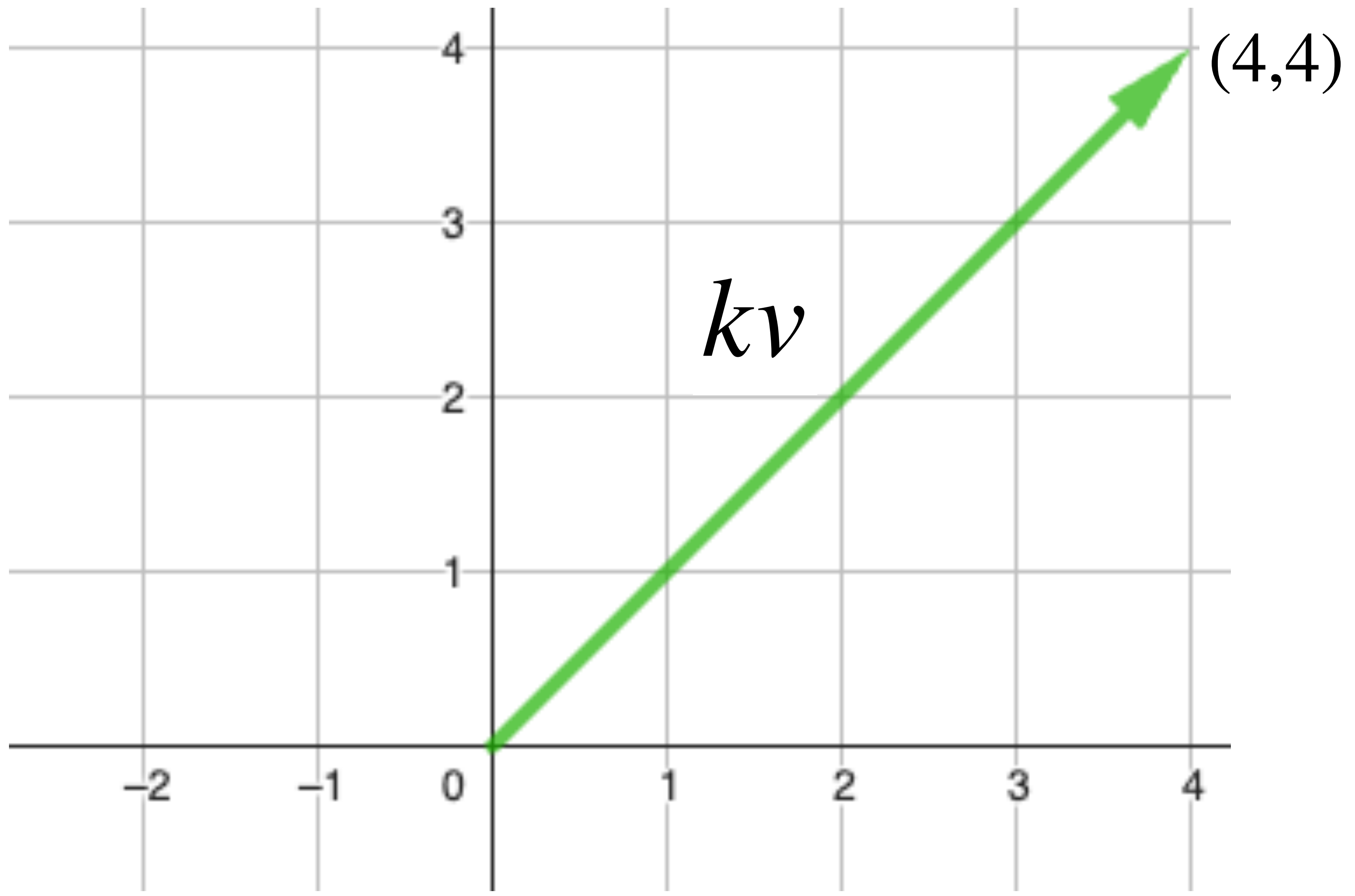
Scalar Vector Multiplication

Multiplying a Vector by a Positive Scalar scales the magnitude for the same direction

$$v = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \qquad k = 2 \qquad \Rightarrow kv = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

$$v = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \cdots \\ x_n \end{bmatrix} \quad \Rightarrow kv = \begin{bmatrix} kx_1 \\ kx_2 \\ kx_3 \\ \cdots \\ kx_n \end{bmatrix}$$



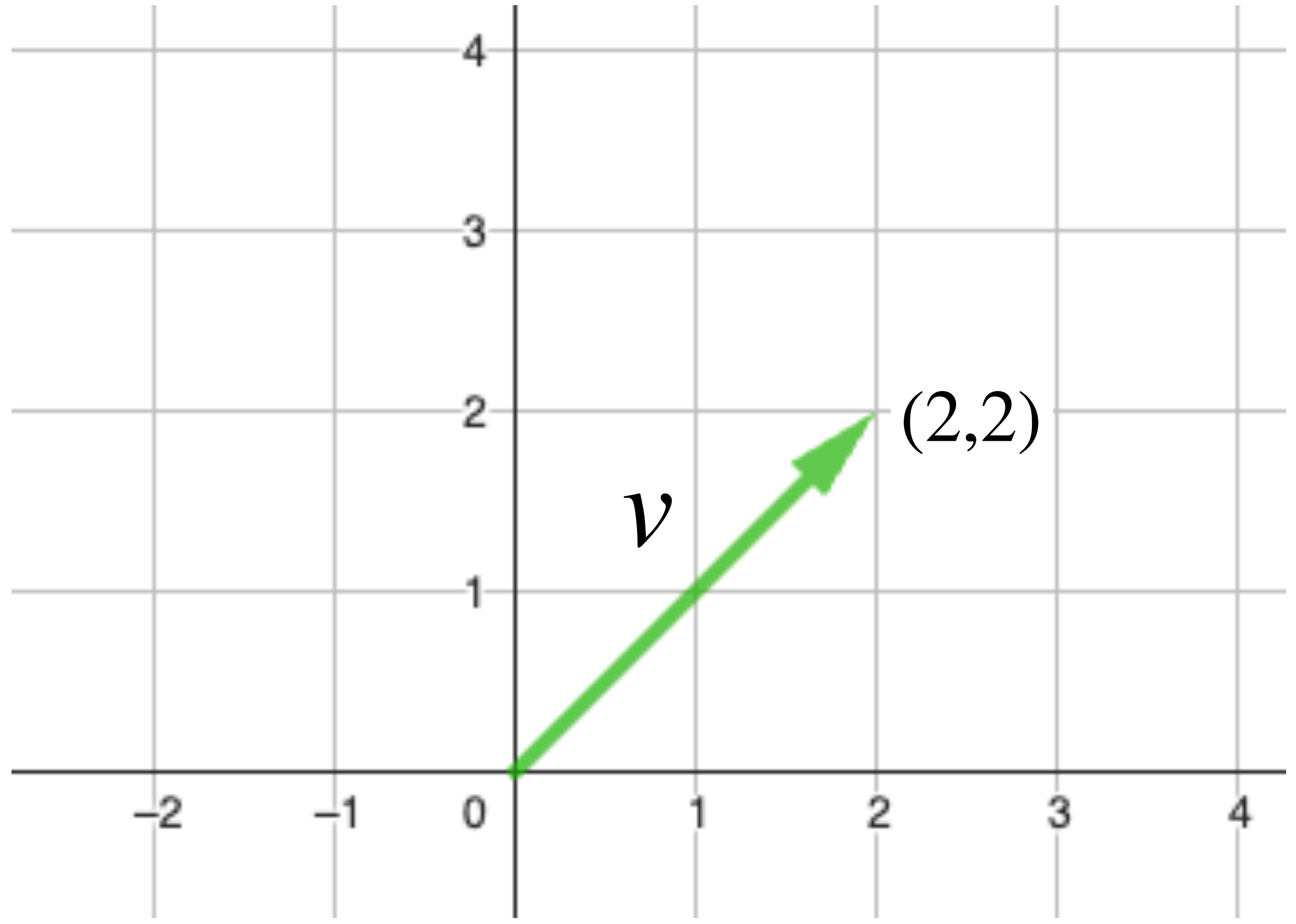


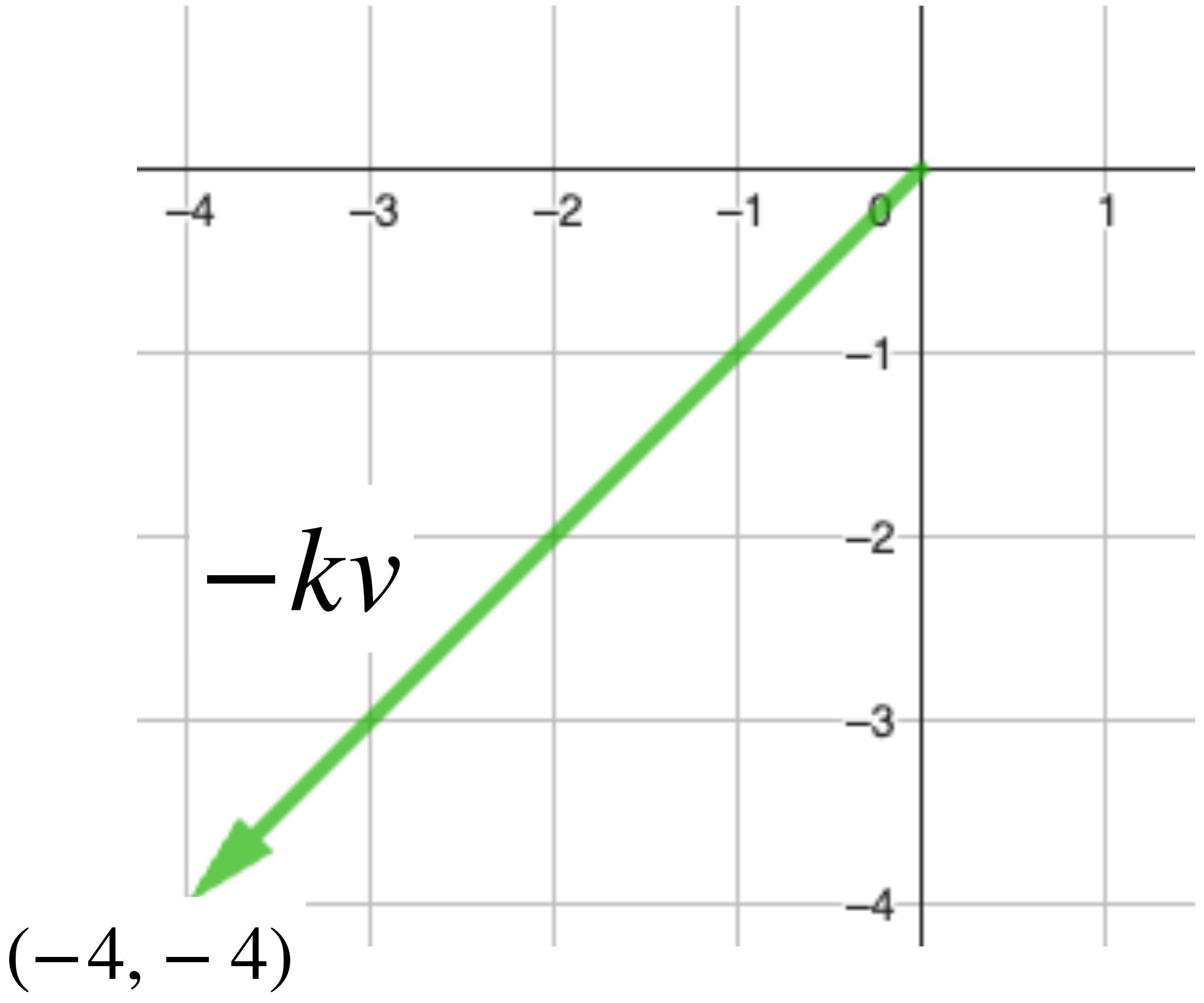
Scalar Vector Multiplication

Multiplying a Vector by a Negative Scalar scales the magnitude for the opposite direction

$$v = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \qquad k = -2 \qquad \Rightarrow kv = \begin{bmatrix} -4 \\ -4 \end{bmatrix}$$

$$v = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \dots \\ x_n \end{bmatrix} \Rightarrow -kv = \begin{bmatrix} -kx_1 \\ -kx_2 \\ -kx_3 \\ \dots \\ -kx_n \end{bmatrix}$$





Scalar Product of Two Vectors

The **Scalar Product of Two Vectors,** also known as the **Dot Product**, results in a Scalar

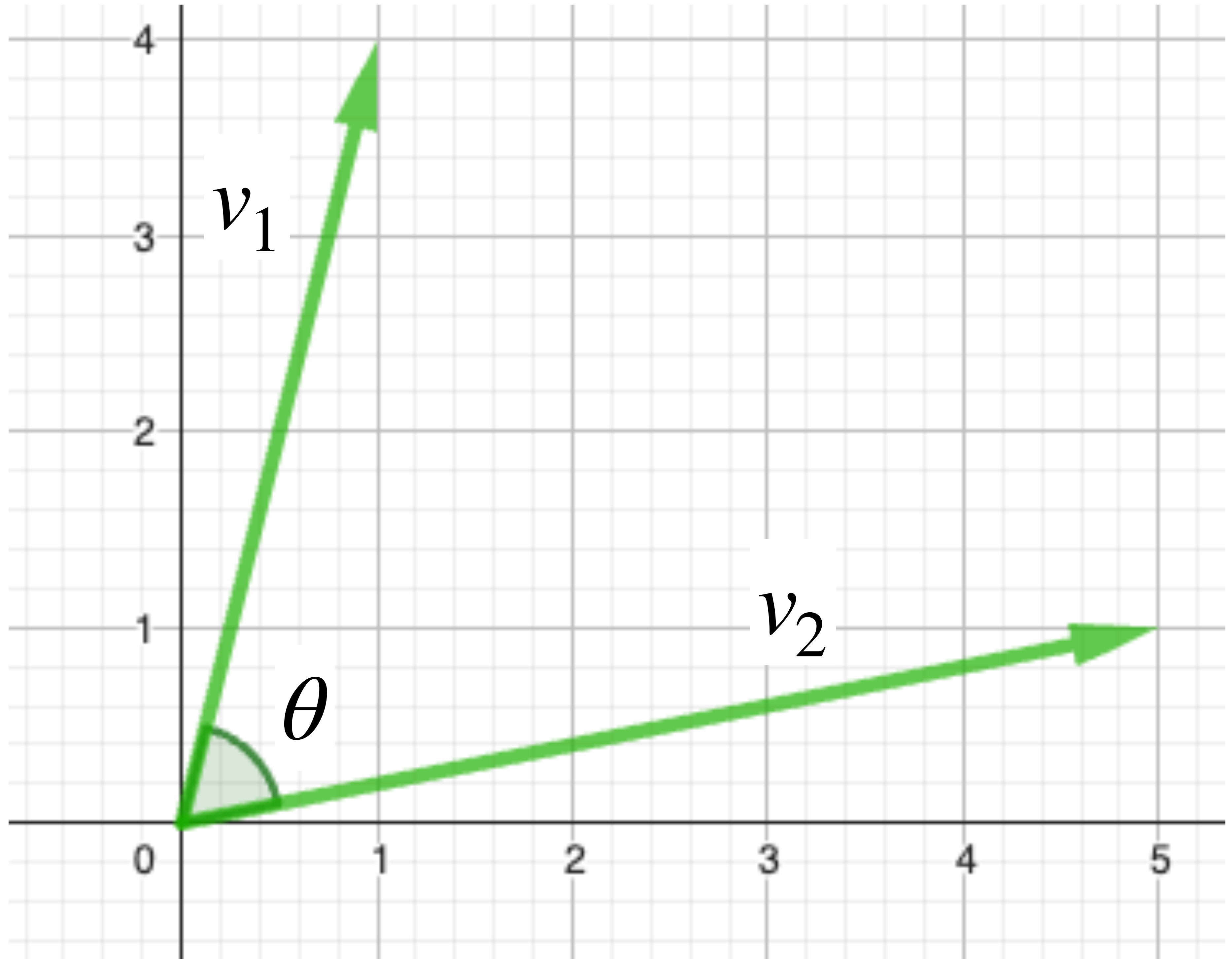
$$v_1 \cdot v_2 = \| v_1 \| \| v_2 \| \; cos\theta$$

$$v_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \quad v_2 = \begin{bmatrix} 5 \\ 1 \end{bmatrix} \quad \theta = 64.65^o$$

$$\| v_1 \| = \sqrt{1^2 + 4^2} = \sqrt{17}$$

$$\| v_2 \| = \sqrt{5^2 + 1^2} = \sqrt{26}$$

$$v_1 \cdot v_2 = \sqrt{17} \times \sqrt{26} \times cos(64.65) = 9$$



Scalar Product of Two Vectors

The **Scalar Product of Two Vectors,** also known as the **Dot Product**, results in a Scalar

$$v_1 = \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \\ \dots \\ x_{1n} \end{bmatrix} \quad v_2 = \begin{bmatrix} x_{21} \\ x_{22} \\ x_{23} \\ \dots \\ x_{2n} \end{bmatrix}$$

$$v_1 \cdot v_2 = x_{11}\,x_{21} + x_{12}\,x_{22} + x_{13}\,x_{23} \dots x_{1n}\,x_{2n}$$

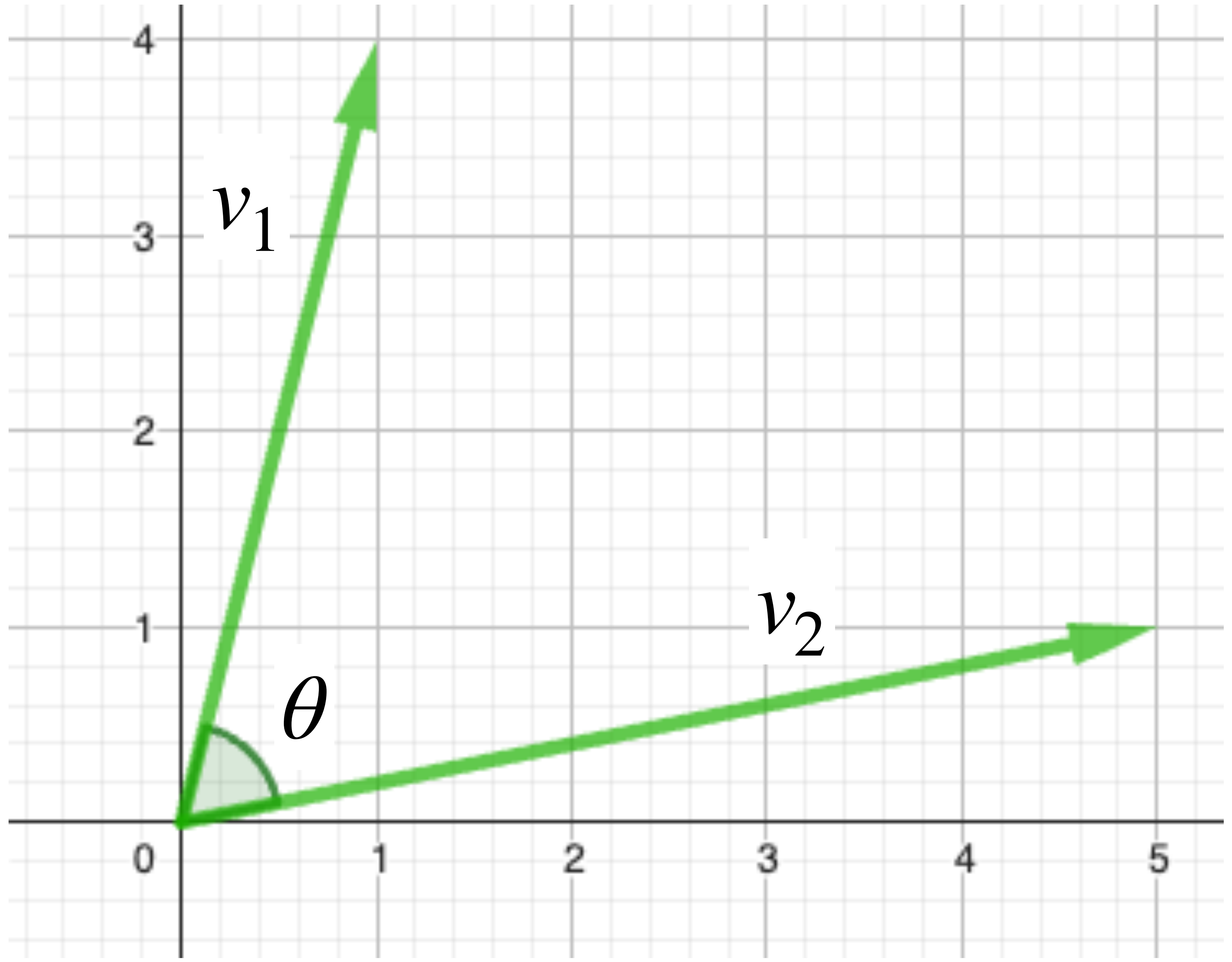
Angle between two Vectors

$$cos(\theta) = \frac{v_1 \cdot v_2}{\parallel v_1 \parallel \parallel v_2 \parallel}$$

Example

$$v_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \quad v_2 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

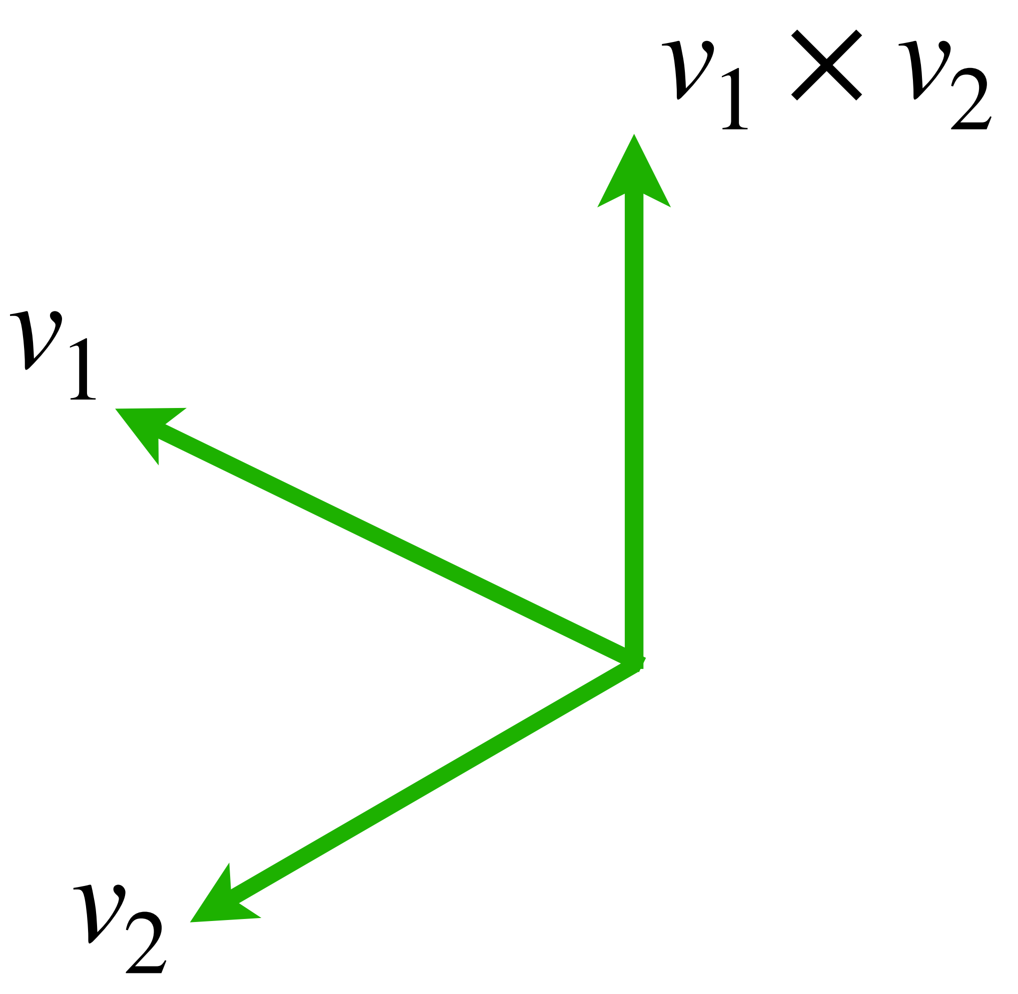
$$v_1 \cdot v_2 = (1 \times 5) + (4 \times 1) = 9$$



Vector Product of Two Vectors

The **Vector Product of Two Vectors,** also known as the **Cross Product**, results in a Vector

$$v_1 \times v_2 = \| v_1 \| \| v_2 \| \, sin(\theta) \, \hat{e}$$



Magnitude is $\| v_1 \| \| v_2 \| \, sin(\theta)$
Direction is given by the unit vector $\hat{e}$
$\hat{e}$ is perpendicular to the plane containing v_1 and v_2
Direction of $\hat{e}$ is given by the right hand rule

$$a = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \quad b = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} \quad \Rightarrow a \times b = \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix}$$

Matrix Determinants are a more convenient method to compute the Cross Product of Two Vectors

While the Vector Cross Product can be generalized to higher dimensions, most applications are in three dimensions

Matrix

A **Matrix** is a rectangular array of numbers, arranged in rows and columns

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

(2×3) matrix

M is a "two by three" matrix

2 rows, 3 columns

An $(n \times 1)$ matrix is a Column Vector

$$v = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \dots \\ x_n \end{bmatrix}$$

An $(1 \times n)$ matrix is a Row Vector

$$v = \begin{bmatrix} x_1 & x_2 & x_3 & \dots & x_n \end{bmatrix}$$

A **Matrix** with the same number of rows and columns is called a square matrix

$$M_1 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

(3×3) matrix

M_1 is a "three by three" matrix

3 rows, 3 columns

$$M_2 = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & 20 \\ 21 & 22 & 23 & 24 & 25 \end{bmatrix}$$

(5×5) matrix

M_2 is a "five by five" matrix

5 rows, 5 columns

The **Identity Matrix** is one where the elements on the diagonal are 1 and the rest are 0

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

An Identity Matrix is always a square matrix

(3×3) identity matrix

The **Zero Matrix** is one where all the elements are zero

$$M = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

A Zero Matrix can have any dimensions

(2×3) zero matrix

Matrix Transpose

The **Matrix Transpose** is an operation that swaps the rows and columns of a Matrix

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \Rightarrow M^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

(2×3) matrix $\quad$ (3×2) matrix

Transpose of a Matrix is a Matrix where the rows become columns and the columns become rows

Transpose of a Square Matrix is another Square Matrix

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \Rightarrow M^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

(3×3) matrix $\quad$ (3×3) matrix

Properties of the Transpose

$$(A + B)^T = A^T + B^T$$
$$(A - B)^T = A^T - B^T$$

$$(A^T)^T = A$$

Matrix Equality

Two Matrices M_1 and M_2 are equal if they have the same number of rows and columns and each element of M_1 is equal to the corresponding element of M_2

$$M_1 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad M_2 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \Rightarrow M_1 = M_2$$

(2×3) matrix $\quad$ (2×3) matrix

Symmetric Matrix

A Matrix M is symmetric if the transpose of the matrix is equal to the original matrix

$$M = \begin{bmatrix} 1 & -2 & -7 \\ -2 & 5 & 6 \\ -7 & 6 & -9 \end{bmatrix} \qquad M^T = \begin{bmatrix} 1 & -2 & -7 \\ -2 & 5 & 6 \\ -7 & 6 & -9 \end{bmatrix}$$

(3×3) matrix $\qquad$ (3×3) matrix

$\boxed{\text{$M$ is a Symmetric Matrix because } M = M^T}$

$\boxed{\text{Only Square Matrices can be Symmetric}}$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

(3×3) matrix $\qquad$ (3×3) matrix

$\boxed{\text{$A$ is not a Symmetric Matrix because } A \neq A^T}$

Matrix Addition of two matrices A and B is defined as the operation of adding the elements of A with the corresponding elements of B

Matrix Addition is only defined for matrices that have the same number of rows and columns

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 8 \\ 9 & 7 \\ 4 & 5 \end{bmatrix} \qquad \Rightarrow A + B = \begin{bmatrix} 1+3 & 4+8 \\ 2+9 & 5+7 \\ 3+4 & 6+5 \end{bmatrix} = \begin{bmatrix} 4 & 12 \\ 11 & 12 \\ 7 & 11 \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & \dots & b_{1n} \\ b_{21} & b_{22} & b_{23} & \dots & b_{2n} \\ b_{31} & b_{32} & b_{33} & \dots & b_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & b_{m3} & \dots & b_{mn} \end{bmatrix}$$

$$A + B = C$$

$$\Rightarrow c_{ij} = a_{ij} + b_{ij}$$

Matrix Addition

Matrix Addition of two matrices A and B is defined as the operation of adding the elements of A with the corresponding elements of B

Matrix Addition is only defined for matrices that have the same number of rows and columns

Matrix Addition is Commutative

$$A + B = B + A$$

Matrix Addition is Associative

$$A + (B + C) = (A + B) + C$$

$A + B$ is a matrix with the same number of rows and columns of A and B

Matrix Subtraction of two matrices A and B is defined as the operation of subtracting the elements of A from the corresponding elements of B

Matrix Subtraction is only defined for matrices that have the same number of rows and columns

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 8 \\ 9 & 7 \\ 4 & 5 \end{bmatrix} \qquad \Rightarrow A - B = \begin{bmatrix} 1-3 & 4-8 \\ 2-9 & 5-7 \\ 3-4 & 6-5 \end{bmatrix} = \begin{bmatrix} -2 & -4 \\ -7 & -2 \\ -1 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & \dots & b_{1n} \\ b_{21} & b_{22} & b_{23} & \dots & b_{2n} \\ b_{31} & b_{32} & b_{33} & \dots & b_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & b_{m3} & \dots & b_{mn} \end{bmatrix}$$

$$A - B = C$$

$$\Rightarrow c_{ij} = a_{ij} - b_{ij}$$

Matrix Subtraction

Matrix Subtraction of two matrices A and B is defined as the operation of subtracting the elements of A from the corresponding elements of B

Matrix Subtraction is only defined for matrices that have the same number of rows and columns

Matrix Subtraction is not Commutative

$$A - B \neq B - A$$

Matrix Subtraction is not Associative

$$A - (B - C) \neq (A - B) - C$$

$A - B$ is a matrix with the same number of rows and columns of A and B

Product of a Scalar and a Matrix

Product of a Scalar s and Matrix A is defined defined as the operation of multiplying the scalar with every element of A

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad s = 3 \qquad \Rightarrow sA = 3 \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 9 \\ 12 & 15 & 18 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad s = -3 \qquad \Rightarrow sA = -3 \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} -3 & -6 & -9 \\ -12 & -15 & -18 \end{bmatrix}$$

Product of a Scalar and a Matrix is distributive

$$s(A + B) = sA + sB$$

$$s(A - B) = sA - sB$$

sA is a matrix with the same number of rows and columns of A

Conjugate Transpose

The **Conjugate Transpose** of a matrix, also known as the Hermitian Transpose, is the operation of applying the complex conjugate to each element followed by the transpose

$$A = \begin{bmatrix} 2+3i & 1+4i \\ -2i & 4 \end{bmatrix}$$

Complex Conjugate of a complex Matrix

$$\bar{A} = \begin{bmatrix} 2-3i & 1-4i \\ 2i & 4 \end{bmatrix}$$

Complex Conjugate of a complex number results in a complex number with the same real and imaginary parts but with the opposite sign

$$\bar{A}^H = \begin{bmatrix} 2-3i & 2i \\ 1-4i & 4 \end{bmatrix}$$

Conjugate Transpose

The **Conjugate Transpose** of a matrix, also known as the Hermitian Transpose, is the operation of applying the complex conjugate to each element followed by the transpose

$$A = \begin{bmatrix} 2+3i & 1+4i \\ -2i & 4 \end{bmatrix}$$

Complex Conjugate of a complex Matrix

$$\bar{A} = \begin{bmatrix} 2-3i & 1-4i \\ 2i & 4 \end{bmatrix}$$

Transpose gives us the Conjugate Transpose

$$\bar{A}^H = \begin{bmatrix} 2-3i & 2i \\ 1-4i & 4 \end{bmatrix}$$

The Conjugate Transpose of a Real Matrix is simply the Transpose

Conjugate Transpose

The **Conjugate Transpose** of a matrix, also known as the Hermitian Transpose, is the operation of applying the complex conjugate to each element followed by the transpose

$$A = \begin{bmatrix} 2 & 4 \\ -2 & 1 \end{bmatrix}$$

$$\bar{A}^H = A^T = \begin{bmatrix} 2 & -2 \\ 4 & 1 \end{bmatrix}$$

The Conjugate Transpose of a Real Matrix is simply the Transpose

Hermitian Matrix

A Matrix that is equal to its Conjugate Transpose (or Hermitian Transpose), is known as a Hermitian Matrix

$$A = \begin{bmatrix} 1 & 2+3i & 4-6i \\ 2-3i & 4 & -2i \\ 4+6i & 2i & 8 \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} 1 & 2-3i & 4+6i \\ 2+3i & 4 & 2i \\ 4-6i & -2i & 8 \end{bmatrix}$$

$$\bar{A}^H = \begin{bmatrix} 1 & 2+3i & 4-6i \\ 2-3i & 4 & -2i \\ 4+6i & 2i & 8 \end{bmatrix}$$

A is a Hermitian Matrix because $A = \bar{A}^H$

The elements on the diagonal must be real because they must be equal to their complex conjugate

Matrix Norm

Matrix Norm is a numeric quantity that gives a measure of the magnitude of a Matrix

1-Norm of a matrix is the maximum of the sum of the absolute values of the columns

$$\| A \|_1 = \max_{1 \le j \le n} \left(\sum_{i=1}^{n} | a_{ij} | \right)$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$\begin{aligned} \| A \|_1 &= max((1+2+3),(4+5+6)) \\ &= max(6,15) \\ &= 15 \end{aligned}$$

Matrix Norm

Matrix Norm is a numeric quantity that gives a measure of the magnitude of a Matrix

Infinity-Norm of a matrix is the maximum of the sum of the absolute values of the rows

$$\| A \|_\infty = \max_{1 \le i \le n} \left(\sum_{j=1}^{n} |a_{ij}| \right)$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$\| A \|_\infty = max((1+4),(2+5),(3+6))$$

$$= max(5,7,9)$$

$$= 9$$

Matrix Norm

Matrix Norm is a numeric quantity that gives a measure of the magnitude of a Matrix

Euclidean Norm of a matrix square root of the sum of squares of the elements

$$\| A \|_E = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} (a_{ij})}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$\| A \|_E = \sqrt{1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2}$$

$$= \sqrt{91}$$

$$= 9.53$$

$$\| A \|_E = \sqrt{A^T A}$$

Matrix Multiplication of two matrices A and B is defined as the operation of multiplying the i^{th} row A to the j^{th} column of B to obtain the element in the i^{th} row and j^{th} column of C

Matrix Multiplication is only defined if the number of columns of A equals the number of rows of C

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 8 \\ 9 & 7 \\ 4 & 5 \end{bmatrix} \quad C = A \times B = \begin{bmatrix} [1 \quad 2 \quad 3] \cdot \begin{bmatrix} 3 \\ 9 \\ 4 \end{bmatrix} & [1 \quad 2 \quad 3] \cdot \begin{bmatrix} 8 \\ 7 \\ 5 \end{bmatrix} \\ [4 \quad 5 \quad 6] \cdot \begin{bmatrix} 3 \\ 9 \\ 4 \end{bmatrix} & [4 \quad 5 \quad 6] \cdot \begin{bmatrix} 8 \\ 7 \\ 5 \end{bmatrix} \end{bmatrix}$$

(2×3) matrix $\quad (3 \times 2)$ matrix

$$\Rightarrow C = A \times B = \begin{bmatrix} (1 \times 3 + 2 \times 9 + 3 \times 4) & (1 \times 8 + 2 \times 7 + 3 \times 5) \\ (4 \times 3 + 5 \times 9 + 6 \times 4) & (4 \times 8 + 5 \times 7 + 6 \times 5) \end{bmatrix} = \begin{bmatrix} 33 & 37 \\ 81 & 97 \end{bmatrix}$$

(2×2) matrix

Matrix Multiplication of two matrices A and B is defined as the operation of multiplying the i^{th} row A to the j^{th} column of B to obtain the element in the i^{th} row and j^{th} column of C

If A is an $m \times n$ matrix and B is an $n \times k$ matrix then $C = AB$ is an $m \times k$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & \dots & b_{1k} \\ b_{21} & b_{22} & b_{23} & \dots & b_{2k} \\ b_{31} & b_{32} & b_{33} & \dots & b_{3k} \\ \vdots & \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & b_{n3} & \dots & b_{nk} \end{bmatrix} \quad C = AB = \begin{bmatrix} c_{11} & c_{12} & c_{13} & \dots & c_{1k} \\ c_{21} & c_{22} & c_{23} & \dots & c_{2k} \\ c_{31} & c_{32} & c_{33} & \dots & c_{3k} \\ \vdots & \vdots & \vdots & & \vdots \\ c_{m1} & c_{m2} & c_{m3} & \dots & c_{mk} \end{bmatrix}$$

where…

$$cij = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

$\forall i = 1\,..\,m$ and $j = 1\,..\,k$

Rules of Matrix Multiplication

Matrix Multiplication is Associative

$$A(BC) = (AB)C$$

Matrix Multiplication is Distributive

$$A(B + C) = AB + AC$$

Matrix Multiplication is not Commutative

$$AB \neq BA$$

Multiplication with the Identity Matrix

$$AI = IA = A$$

Multiplication with the Zero Matrix

$$OA = AO = O$$

Multiplying an $m \times n$ matrix with an $n \times k$ matrix produces an $m \times k$ matrix

Multiplication of an $m \times n$ matrix with an $p \times q$ matrix, where $n \neq q$ is undefined

Determinant of a Matrix

Determinant of a matrix A, denoted by $|a|$ or $det(A)$, is function that returns a scalar value computed from the elements of the matrix

Determinant of a 2×2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = |a_{11}a_{22} - a_{21}a_{12}|$$

Example

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = |2 \times 5 - 4 \times 3| = |10 - 12| = -2$$

Determinants are only defined for square matrices

Determinant of a Matrix

Determinant of a matrix A, denoted by $|a|$ or $det(A)$, is function that returns a scalar value computed from the elements of the matrix

Cofactor Expansion (aka Laplace Expansion) to calculate the determinant

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Every element of this 3×3 matrix has an associated minor

Determinant of a Matrix

Determinant of a matrix A, denoted by $|a|$ or $det(A)$, is function that returns a scalar value computed from the elements of the matrix

Cofactor Expansion (aka Laplace Expansion) to calculate the determinant

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & \boxed{5} & \boxed{6} \\ 7 & \boxed{8} & \boxed{9} \end{bmatrix}$$

Every element of this 3×3 matrix has an associated minor

The associated minor for a_{11} (represented by A_{11}) is the determinant of the 2×2 matrix formed by removing the row 1 and column 1

Associated minor for a_{11}

$$A_{11} = \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} = |(5 \times 9) - (8 \times 6)| = |45 - 48| = 3$$

Determinant of a Matrix

Determinant of a matrix A, denoted by $|a|$ or $det(A)$, is function that returns a scalar value computed from the elements of the matrix

Cofactor Expansion (aka Laplace Expansion) to calculate the determinant

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Every element of this 3×3 matrix has an associated minor

The associated minor for a_{12} (represented by A_{12}) is the determinant of the 2×2 matrix formed by removing the row 1 and column 2

Associated minor for a_{12}

$$A_{12} = \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} = |(4 \times 9) - (7 \times 6)| = |36 - 42| = 6$$

Determinant of a Matrix

Determinant of a matrix A, denoted by $|a|$ or $det(A)$, is function that returns a scalar value computed from the elements of the matrix

Cofactor Expansion (aka Laplace Expansion) to calculate the determinant

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Every element of this 3×3 matrix has an associated minor

The associated minor for a_{13} (represented by A_{13}) is the determinant of the 2×2 matrix formed by removing the row 1 and column 3

Associated minor for a_{13}

$$A_{13} = \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = |(4 \times 8) - (7 \times 5)| = |32 - 35| = 3$$

Determinant of a Matrix

Determinant of a matrix A, denoted by $|a|$ or $det(A)$, is function that returns a scalar value computed from the elements of the matrix

Cofactor Expansion (aka Laplace Expansion) to calculate the determinant

$$A = \begin{bmatrix} 1 & \boxed{2 \quad 3} \\ 4 & 5 & 6 \\ 7 & \boxed{8 \quad 9} \end{bmatrix}$$

Every element of this 3×3 matrix has an associated minor

The associated minor for a_{21} (represented by A_{21}) is the determinant of the 2×2 matrix formed by removing the row 2 and column 1

Associated minor for a_{21}

$$A_{21} = \begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} = |(2 \times 9) - (8 \times 3)| = |18 - 24| = 6$$

Determinant of a Matrix

Determinant of a matrix A, denoted by $|a|$ or $det(A)$, is function that returns a scalar value computed from the elements of the matrix

Cofactor Expansion (aka Laplace Expansion) to calculate the determinant

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

The Determinant is calculated by the linear combination of the product of the elements of the first row and their associated minors with alternating signs

Alternating '+' and '-' signs

$$|A| = + a_{11}A_{11} - a_{12}A_{12} + a_{13}A_{13}$$

Determinant of a matrix A, denoted by $|a|$ or $det(A)$, is function that returns a scalar value computed from the elements of the matrix

Cofactor Expansion (aka Laplace Expansion) to calculate the determinant

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & \boxed{5} & \boxed{6} \\ 7 & \boxed{8} & \boxed{9} \end{bmatrix}$$

The Determinant is calculated by the linear combination of the product of the elements of the first row and their associated minors with alternating signs

Start with a '+' sign

$$|A| = +1 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix}$$

Determinant of a Matrix

Determinant of a matrix A, denoted by $|a|$ or $det(A)$, is function that returns a scalar value computed from the elements of the matrix

Cofactor Expansion (aka Laplace Expansion) to calculate the determinant

$$A = \begin{bmatrix} 1 & 2 & 3 \\ \boxed{4} & 5 & \boxed{6} \\ \boxed{7} & 8 & \boxed{9} \end{bmatrix}$$

The Determinant is calculated by the linear combination of the product of the elements of the first row and their associated minors with alternating signs

Alternate a '-' sign

$$|A| = +1 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix}$$

Determinant of a Matrix

Determinant of a matrix A, denoted by $|a|$ or $det(A)$, is function that returns a scalar value computed from the elements of the matrix

Cofactor Expansion (aka Laplace Expansion) to calculate the determinant

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

The Determinant is calculated by the linear combination of the product of the elements of the first row and their associated minors with alternating signs

Alternate a '+' sign

$$|A| = +1 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

Determinant of a Matrix

Determinant of a matrix A, denoted by $|a|$ or $det(A)$, is function that returns a scalar value computed from the elements of the matrix

Cofactor Expansion (aka Laplace Expansion) to calculate the determinant

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

The Determinant is calculated by the linear combination of the product of the elements of the first row and their associated minors with alternating signs

$$|A| = +1 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= 1|(5 \times 9) - (8 \times 6)| - 2|(4 \times 9) - (7 \times 6)| + 3|(4 \times 8) - (7 \times 5)|$$

$$= 1|45 - 48| - 2|36 - 42| + 3|32 - 35| = (1 \times 3) - (2 \times 6) + (3 \times 3)$$

$$= 3 - 12 + 9 = 0$$

Inverse of a Matrix

Inverse of a matrix A, denoted by A^{-1} is a matrix such that when multiplied by the original results in the Identity Matrix.

$$AA^{-1} = A^{-1}A = I$$

The inverse of a matrix is undefined for a non-square matrix
The inverse of a matrix is undefined for some square matrices

If a matrix has an inverse then it is said to be non-singular
If a matrix does not have an inverse then it is said to be singular

Inverse of a 2×2 matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad A^{-1} = \frac{1}{|A|}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} = \frac{1}{|a_{11}a_{22} - a_{21}a_{12}|}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Inverse of a matrix A, denoted by A^{-1} is a matrix such that when multiplied by the original results in the Identity Matrix.

$$AA^{-1} = A^{-1}A = I$$

If the determinant is zero then the inverse of the matrix is undefined

Inverse of a 2×2 matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad A^{-1} = \frac{1}{|A|} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} = \frac{1}{|a_{11}a_{22} - a_{21}a_{12}|} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Inverse of a Matrix

Inverse of a matrix A, denoted by A^{-1} is a matrix such that when multiplied by the original results in the Identity Matrix.

$$AA^{-1} = A^{-1}A = I$$

Inverse of a 2×2 matrix is calculated by swapping the diagonal elements, reversing the sign of the other two and then dividing by the determinant of A

Inverse of a 2×2 matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} = \frac{1}{|a_{11}a_{22} - a_{21}a_{12}|}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Inverse of a Matrix

Inverse of a matrix A, denoted by A^{-1} is a matrix such that when multiplied by the original results in the Identity Matrix.

$$AA^{-1} = A^{-1}A = I$$

Inverse of a 2×2 matrix is calculated by swapping the diagonal elements, reversing the sign of the other two and then dividing by the determinant of A

Inverse of a 2×2 matrix:

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} 3 & -0 \\ -2 & 1 \end{bmatrix} = \frac{1}{|1 \times 3 - 2 \times 0|}\begin{bmatrix} 3 & 0 \\ -2 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3 & 0 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{-2}{3} & \frac{1}{3} \end{bmatrix}$$

Inverse of a Matrix

Inverse of a matrix A, denoted by A^{-1} is a matrix such that when multiplied by the original results in the Identity Matrix.

$$AA^{-1} = A^{-1}A = I$$

Inverse of a 2×2 matrix:

Inverse of a 2×2 matrix is calculated by swapping the diagonal elements, reversing the sign of the other two and then dividing by the determinant of A

$$A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} = \frac{1}{|(1 \times 2) - (-2 \times -1)|}\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} = \frac{1}{|2-2|}\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} = \frac{1}{0}\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$$

Determinant of A is zero.
A^{-1} is undefined

Related Tutorials & Textbooks

[Logistic Regression]

An introduction to Logistic Regression. A Logistic Regression model use used to predict a binary value (the dependent variable) for one or more independent variables using a threshold to classify a probability.

[Multiple Regression]

Multiple regression extends the two dimensional linear model introduced in Simple Linear Regression to k + 1 dimensions with one dependent variable, k independent variables and k+1 parameters.

[Cost Function & Gradient Descent for Logistic Regression]

An introduction to the Cost function for Logistic Regression long with its partial derivative (the gradient vector). The model parameters (B & W) are then optimized using Maximum Likelihood Estimation and Gradient Descent.

For a complete list of tutorials see:
https://arrsingh.com/ai-tutorials